DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MASTER'S OPTION EXAM-APPLIED MATHEMATICS
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Do five of the following problems. All problems carry equal weight.
Passing level: 60% with at least two substantially correct.

1. Solve the diffusion problem $u_t = ku_{xx}$ in $0 < x < l$, with mixed boundary conditions $u(0,t) = u_x(l,t) = 0$.

2. Consider the ordinary differential equation

$$x' = x^2 - 3x + 2$$

   (a) What are the constant solutions?

   (b) Sketch the solutions of the ODE with initial data $x(0) = -2, .5, 3$ and find $\lim_{t\to\pm\infty} x(t)$ for each solution. Do not solve the equation!

3. Consider the system

$$\frac{dx}{dt} = x + y^2$$
$$\frac{dy}{dt} = x + y$$

   (a) Determine all critical points of the system.

   (b) Find the corresponding linear system near each critical point.

   (c) Discuss the stability of the solution near each critical point.

4. (a) Give a physical interpretation of the equation

$$u_t + 2x^2 u_x = 0$$

   (b) Draw the characteristics and solve the above equation with initial data $u(x, 0) = e^x$.

5. Consider the eigenvalue problem with Robin Boundary Conditions at both ends:

$$-X'' = \lambda X$$

$$X'(0) - a_0 X(0) = 0, X'(l) + a_l X(l) = 0$$

   (a) Show that $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_l = -a_0 a_l l$.

   (b) Find the eigenfunctions corresponding to the zero eigenvalue.

6. Consider an infinite string whose position is governed by the 1-D wave equation

$$u_{tt} = u_{xx}$$

The string's initial position $u(x, 0) = 0$ and initial velocity $u_t(x, 0) = 1$ for $|x| > a$ and $u_t(x, 0) = 1$ for $|x| \geq a$. Sketch the string profile (u versus x) at each of the successive instants $t = a/2, 3a/2$ and $5a$. [Hint: Use D'Alembert's solution].

7. (a) Use separation of variables to solve Laplace's equation $\Delta u = 0$ with 'initial data'

$$u^n(x, 0) = \frac{1}{n}\sin(nx) \quad \text{and} \quad \frac{\partial u^n}{\partial t}(x, 0) = 0.$$

(b) Show that the initial value problem for Laplace's equation is ill–posed by considering the limit of data and solution as $n \to \infty$.