Do five of the following problems. All problems carry equal weight. Passing level: 75% with at least three substantially complete solutions.

1a) Find an energy function for the system
\[
\begin{align*}
x' &= y \\
y' &= x - x^4
\end{align*}
\]
and use this to sketch the phase plane of (1).

1b) Show that for sufficiently small $\varepsilon > 0$ the system of differential equations
\[
\begin{align*}
x' &= y \\
y' &= x - x^4 + \varepsilon \sin t
\end{align*}
\]
has a periodic solution $(x_\varepsilon(t), y_\varepsilon(t))$ with period $2\pi$ that remains in a neighborhood of the origin. (Hint: augment the equations with $\tau' = 1, \tau(0) = 0$ so that $\tau(t) = t$ and rewrite the problem as a fixed point problem for a map.

2) Assume that $u$ is harmonic in $\Omega$.

a) Let $\varphi_\varepsilon$ be a standard mollifier; show that
\[
u'(x) := (\varphi_\varepsilon * u)(x) = u(x)
\]
for all $x \in \Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial \Omega) > \varepsilon\}$ (Hint: use the mean value property)

b) Show that $u \in C^\infty(\Omega)$. 
3a) Find a function $K(x, y)$ such that the solutions of the inhomogeneous linear boundary value problem

\[
\begin{align*}
    u'' &= -u + h(x), \quad 0 < x < L \\
    u(0) &= 0, \quad u'(L) = u(L)
\end{align*}
\]

are the integrals $u(x) = \int_0^1 K(x, y)h(y) \, dy$.

3b) Using the representation in 3a) and the method of successive approximations, show that if $L$ is sufficiently small and $f(u)$ is a given smooth function of $u$, the nonlinear boundary value problem

\[
\begin{align*}
    u'' &= -u + f(u(x)), \quad 0 < x < L \\
    u(0) &= 0, \quad u'(L) = u(L)
\end{align*}
\]

has a continuous solution $u(x)$. Explain why $u$ has two continuous derivatives.

4) Suppose $u \in \mathcal{S}(\mathbb{R}^n)$, where $u = u(x), x = (y, z)$ and $y \in \mathbb{R}^k, z \in \mathbb{R}^{n-k}$. Define the (trace) map

\[
T : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^k)
\]

as $(Tu)(y) = u(y, 0)$.

Show that $T$ can be extended to a bounded linear map $T : H^t(\mathbb{R}^n) \longrightarrow H^s(\mathbb{R}^k)$, provided $s < t - \frac{n-k}{2}$.

**Hint:** First show that for all $u \in \mathcal{S}(\mathbb{R}^n)$

\[
\| Tu \|_{H^s(\mathbb{R}^k)} \leq C(n, k) \| u \|_{H^t(\mathbb{R}^n)}
\]

where $C(n, k)$ is a constant depending only on $k, n$. Also recall that

\[
H^s(\mathbb{R}^k) = \{ u \in L^2(\mathbb{R}^k) : (1 + |\zeta|^2)^{s/2} \hat{u}(\zeta) \in L^2(\mathbb{R}^k) \}\]
5a) Suppose that \( \ell \) is a positive constant, and that \( U(x,t) \) is a smooth, classical solution of the PDE on \([0, 1] \times [0, T]\) of

\[
U_{tt} = U_{xx} - \ell \sin U, \quad 0 < x < 1 \tag{2}
\]

\[
U(0, t) = U(1, t) = 0, \quad 0 \leq t \leq T
\]

\[
(U(x, 0), U_x(0, t)) = (f(x), g(x))
\]

Consider the system of ODE’s

\[
\begin{align*}
    u'_j &= v_j \\
    v'_j &= -\ell \sin u_j + N^2(u_{j+1} - 2u_j + u_{j-1}) + E_j(t),
\end{align*}
\]

for \( 1 \leq j \leq N \), where \( u_j \equiv v_j \equiv 0 \) for \( j = 0 \) and \( j = N + 1 \), and where \( E_j(t) \) are specified functions. Let \( x_j = j/N \) for \( 1 \leq j \leq N \) and define \( u_j(t) = U(x_j, t) \) and \( v_j(t) = U_x(x_j, t) \). Show that \( u_j(t), v_j(t) \) satisfy a system of the form (3) on \( 0 \leq t \leq T \) for some functions \( E_j(t) \) that satisfy the estimate: \( |E_j(t)| \leq K/N^2 \) for some constant \( K > 0 \) depending only on \( U \) and its derivatives up to fourth order.

5b) Now suppose that \( \bar{u}_j(t), \bar{v}_j(t) \) is the solution of the homogeneous system

\[
\begin{align*}
    u'_j &= v_j \\
    v'_j &= -\ell \sin u_j + N^2(u_{j+1} - 2u_j + u_{j-1}),
\end{align*}
\]

obtained as a formal approximation to (2) for large \( N \). Calculate the eigenvalues of the linearization of this system at the rest point \( u_j = v_j = 0 \) for all \( j \). What, if anything, can be concluded from this calculation?

5c) Find an energy function \( E(u, v) \) for the homogeneous system in (4), and use this to show that the rest point in at the origin is stable for all \( N \). (Hint: what does the energy for the PDE look like?)
Suppose \( u \) is a smooth solution of
\[
\begin{cases}
  u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0, T) \\
  u(x, 0) = g(x) & \text{in } \mathbb{R}^n
\end{cases}
\]

Show that if \( u \) and all its spatial derivatives decay at infinity, then
\[
\max_{0 \leq t \leq T} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx + \int_0^T \int_{\mathbb{R}^n} (u_t^2 + |\nabla^2 u|^2) \, dx \, dt \leq C \left[ \int_0^T \int_{\mathbb{R}^n} f(x, t)^2 \, dx \, dt + \int_{\mathbb{R}^n} |\nabla g|^2 \, dx \right]
\]