Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Topology

January 24, 2003

Answer five of seven questions. Indicate clearly which five questions you want to have graded. Justify your answers.

Passing Standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

(1) Consider the letters of the alphabet N, O, P as subsets of the plane made up of straight line segments and arcs of circles. Decide whether any of them are homeomorphic.

(2) Show that a compact subset of a metric space is closed and bounded.

(3) Define an equivalence relation on $X = \mathbb{R}^2 \setminus \{0\}$ by letting $(x, y) \sim (x', y')$ iff both points are on the same connected component of the curve defined by $xy = c$ for some constant $c \in \mathbb{R}$. Show that the quotient space $X/\sim$ is non-compact and non-Hausdorff.

(4) If A is a connected subspace of a topological space X, show that the closure $\overline{A}$ is connected. If A is path connected, is $\overline{A}$ necessarily path-connected?

(5) Suppose the metric space X is separable; i.e. it has a countable dense subset. Show that it is second-countable (it has a countable basis for its topology).

(6) Define a function $f \colon \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ if $x \notin [0, 1]$ and $f(x) = 1 - |2x - 1|$ if $x \in [0, 1]$. Define a sequence $\{g_n\}$ in the space $\mathcal{C}(\mathbb{R})$ of all continuous functions $\mathbb{R} \to \mathbb{R}$ by $g_n(x) = f(nx)$. Determine whether this sequence converges in each of the following topologies: product, compact-open, and uniform.

(7) Suppose X is a metric space and let $K(X)$ be the set of all compact subsets of X. If $A, B \in K(X)$, define

$$D(A, B) = \max\{\max_{x \in A} d(x, B), \max_{y \in B} d(y, A)\},$$

where $d(z, C) = \min_{y \in C} d(z, y)$ for any $z \in X$, $C \in K(X)$.

(a) Show that D is a metric on $K(X)$.
(b) Show that $K(\mathbb{R}^n)$ is path-connected. (Hint: if $A, B \in K(X)$, consider their "join" $A * B$, the union of all line segments in $\mathbb{R}^{n+1}$ joining a point of $A \times \{0\}$ to a point of $B \times \{1\}$; show that $A * B$, thought of as a subset of $K(X) \times \mathbb{R}$, is the graph of a path between A and B.)