

**DEPARTMENT OF MATHEMATICS AND STATISTICS**  
**UNIVERSITY OF MASSACHUSETTS**  
**BASIC EXAM – PROBABILITY**  
**August 27, 2003**

*Work all problems. 60 points are needed to pass at the Master's level and 75 to pass at the Ph.D. level.*

1. (20 pts) Suppose 3 boys and 3 girls stand in a line in random order.
  - (a) What is the probability that the subsequence of boys is ascending in height and the subsequence of girls is ascending in height.
  - (b) What is the probability that the sequence alternates between boys and girls?
  - (c) What is the probability that the three boys are not together?
2. (20 pts) Let  $X \sim N(\mu, \sigma^2)$ .
  - (a) Show the MGF of  $X$  is  $M_X(t) = e^{\mu t + \sigma^2 t^2 / 2}$ .
  - (b) Show that if  $X_1 \sim N(\mu_1, \sigma_1^2)$  and  $X_2 \sim N(\mu_2, \sigma_2^2)$  with  $X_1$  and  $X_2$  independent then  $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$
3. (20 points) Suppose each of 100 genes has probability 0.2 of mutating in a given time period, and the genes act independently. Let  $N$  = the number of the genes which mutate.
  - (a) Derive the exact distribution of  $N$ .
  - (b) State the CLT (central limit theorem) and use it to approximate  $P(N < 10)$ .
4. (20 pts) The daily number of visits to a particular website,  $X$ , is Poisson with parameter  $\Lambda$ .  $\Lambda$  varies independently from day to day according to an exponential distribution:

$$f(\lambda) = 1_{(0, \infty)}(\lambda) c e^{-c\lambda}$$

- (a) Find an integral expression for  $P(X = 3, \Lambda \leq 5)$ . Don't evaluate the integral.
  - (b) What is the distribution of the random variable  $E(X|\Lambda)$
  - (c) Use the preceding part to compute  $E(X)$ .
  - (d) Assuming the number of visits to the website is independent from day to day, how many days would be expected to have exactly 2 visits over the course of a year?
5. (20 pts) Let  $X_1$  and  $X_2$  be independent random variables each having an exponential distribution with mean 1. Let

$$\begin{aligned} Y_1 &= X_1/X_2 \\ Y_2 &= X_2 \end{aligned}$$

- (a) Without any calculation give the marginal distribution of  $Y_2$  and the conditional distribution of  $Y_1$  given  $Y_2 = y_2$ .
- (b) Find the joint pdf of  $Y_1$  and  $Y_2$ .
- (c) Find the marginal pdf of  $Y_1$ .