DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - NUMERICS
January, 2003

Do five of the following problems. All problems carry equal weight.
Passing level:
Masters: 60% with at least two substantially correct.
Ph.D.: 75% with at least three substantially correct.

1. Find the equation of the line that best fits the data

$$(-2,1),(-1,3),(0,2),(1,1)$$

in the least squares sense.

2. In order to solve the equation

$$x^2 = 2$$

one can use the following method

$$x_{n+1} = \sqrt{2 + x_n}$$

Determine an interval $[a, b]$ containing the root such that for each x_0 in $[a, b]$, the iteration will converge to the root. Prove it.

3. Find the Taylor expansion for the function

$$g(x) = \log\left(\frac{1+x}{1-x}\right),$$

and bound the error on the interval $x \in [-1/2, 1/2]$. What is the most efficient way of evaluating this polynomial?

4. Use Newton divided differences to write the secant method as

$$x_{k+1} = x_k - \frac{f(x_k)}{f[x_k, x_{k-1}]},$$

and show that the error satisfies

$$x_{k+1} - \alpha = \frac{f[x_k, x_{k-1}, \alpha]}{f[x_k, x_{k-1}]}(x_k - \alpha)(x_{k-1} - \alpha).$$

5. Define a *P-matrix* to be an $n \times n$ matrix in which $a_{ij} = 0$ if $i + j \leq n$, and a *Q-matrix* to be a P-matrix in which $a_{ij} = 1$ if $i + j = n + 1$.

(a) Find the *PQ-factorization* of the matrix

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 11 & 8 \\ 6 & 20 & 19 \end{pmatrix}.$$

(b) Use this to solve the linear system $A\ x = b$ in an efficient way, where $b = (2, 0, 5)^t$.

6. Find the coefficients A, B and C for the most accurate numerical method of the form

$$y_{n+1} = y_{n-3} + Af(x_n, y_n) + Bf(x_{n-1}, y_{n-1}) + Cf(x_{n-2}, y_{n-2})$$

for the initial value problem $y' = f(x, y)$ with $y(x_0) = y_0$. Find the error term.

7. Let $f(x) = \cos^{-1}(x)$ for $-1 \leq x \leq 1$(the principal branch $0 \leq f \leq \pi$). Find the polynomial of degree one,

$$p(x) = a_0 + a_1 x$$

which minimizes

$$\int_{-1}^{1} \frac{[f(x) - p(x)]^2}{\sqrt{1 - x^2}} dx$$

Hint: Use the Chebyshev polynomials $T_n(x) = \cos(n \cos^{-1} x)$ which are orthogonal with respect to the weight function

$$w(x) = \frac{1}{\sqrt{1 - x^2}}$$