Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Topology

January 23, 2002

Answer five of seven questions. Indicate clearly which five questions you want to have graded. Justify your answers.

Passing Standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

1. Consider these topologies on $\mathbb{R}$:
   (i) discrete (= fine topology),
   (ii) standard (= order topology), and
   (iii) Zariski (= finite complement topology).
   For each topology, determine whether or not the interval $(-\infty, 2002)$ is

   (a) open, (b) closed, (c) compact (in the subspace topology).

2. The *Sorgenfrey plane* S has underlying set $\mathbb{R}^2$ with basic open sets of the form
   $$U = [x, x') \times [y, y').$$
   (a) Is the diagonal line R defined by the equation $x + y = 0$ a closed subset of S?

   (b) Describe the topology induced on R as a subspace of S.

3. Let Q be a countable set of points in $\mathbb{R}^2$. Prove or refute: its complement $\mathbb{R}^2 \setminus Q$ is connected.

4. Let Z be a locally compact Hausdorff space with one-point compactification $\widehat{Z}$. Show that $\widehat{Z}$ is connected if each connected component of Z is nocompact. If Z is also locally connected, show that "if" can be replaced by "if and only if".

5. Let $f : X \to X$ be a continuous map from a compact Hausdorff space X to itself. Define its *mapping torus* as the quotient space
   $$T_f = (X \times [0, 1])/\{(x \times 0) \sim (f(x) \times 1)\}.$$
   (a) Verify that T_f is also a compact Hausdorff space.

   (b) When f is the identity map, T_f is homeomorphic to the product space $X \times S^1$. Give an example of a space X and a homeomorphism $f : X \to X$ such that T_f is *not* homeomorphic to $X \times S^1$.

6. Prove that the topology of a compact metric space has a countable basis; give a noncompact counterexample.

7. Let $\mathcal{P}$ be the set of real polynomials in one variable, which we may regard as a subspace of

$$\mathcal{C} = \{\text{continuous functions } \mathbb{R} \to \mathbb{R}\}$$

with the compact-open topology, or as a subspace of

$$\mathcal{F} = \{\text{all functions } \mathbb{R} \to \mathbb{R}\}$$

with the point-open topology.

(a) Show that the two subspace topologies on $\mathcal{P}$ coincide.

(b) Prove or refute: $\mathcal{P}$ is a closed subset of $\mathcal{C}$ and of $\mathcal{F}$.