DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM – PROBABILITY
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Work all problems. 60 points are sufficient to pass at the Master's level and 75 to pass at the Ph.D. level.

1. (24 pts) Let Q be the unit square in the xy-plane and A a region in Q. Let α be the area of A. Choose n points independently and unifromly distributed over Q. Let

$$X_i = \begin{cases} 1 & \text{if the } i\text{th point lies in } A \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, \ldots, n$.

   (a) Find the expected value, $E(X_i)$, and the variance, $V(X_i)$ for $1 \leq i \leq n$.

   (b) Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. If n is large, $\bar{X}$ is, with high probability, a good approximation to what number? Why?

   (c) Use the CLT to determine how large n should be so that

$$P(|\bar{X} - \alpha| \leq .01) = .99$$

   when $\alpha = .2$.

   (d) Use Chebychev's inequality to give an upper bound for $P(|\bar{X} - \alpha| > .01)$ Does your answer require that n be "large?"

2. (18 pts) Let X have pdf

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

and let Y be the greatest integer less than or equal to X.

   (a) Find the probability distribution of Y.

   (b) Compute $E(Y)$.

   (c) Compute the variance of Y.

3. (18 pts) Let X_1 and X_2 be random variables with joint pdf

$$f(x_1, x_2) = \begin{cases} 4x_1x_2 & \text{for } 0 < x_1 < 1 \text{ and } 0 < x_2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

Let

$$\begin{aligned} Y_1 &= X_1/X_2 \\ Y_2 &= X_1 * X_2 \end{aligned}$$

   (a) Sketch the region S consisting of all points (y_1, y_2) such that $f_{Y_1,Y_2}(y_1, y_2) > 0$.

   (b) Find the joint pdf of Y_1 and Y_2.

   (c) Find the marginal pdf of Y_1.

4. (12 pts) A Poisson random variable with mean μ has pdf

$$f(x) = \frac{e^{-\mu}\mu^x}{x!}$$

for $x = 0, 1, 2, \ldots$

(a) Let X be Possion with mean μ. Compute the moment generating function of X. It may help to remember that

$$e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!}$$

(b) If $X_1, \ldots, X_n$ are independent Poisson variables with means $\mu_1, \ldots, \mu_n$, what is the moment generating function of $Y = \sum_{k=1}^{n} X_k$?

(c) What is the distribution of Y?

5. (18 pts) Let X be a standard normal random variable, and let Y be a random variable such that $E(Y|X = x) = ax + b$, for some known a and b, and $V(Y|X = x) = 1$.

(a) Show that $E(Y) = b$.

(b) Show that $V(Y) = 1 + a^2$.

(c) Show that $E(XY) = a$.

6. (12 pts) Three molecules of type A, three of type B, three of type C, and three of type D are to be linked together to form a chain molecule. One such chain molecule is ABCDABCDABCD.

(a) How many such chain molecules are there?

(b) Suppose all of the different molecule structures are equally likely. What is the probability that all three molecules of each type end up next to each other (as in BBBAAADDDCCC)?