(1) Let $X$ and $Y$ be independent, non-negative, real-valued random variables. Their respective cumulative distribution functions are $F(x)$ and $G(x)$ and their respective probability densities (with respect to Lebesgue measure) are $f(x) > 0$ and $g(x) > 0$. Define $Z = \min(X, Y)$ and 

$$I = 0, \text{ if } X \geq Y; I = 1, \text{ if } X < Y.$$ 

(a) Derive the joint distribution of $(Z, I)$.

(b) Verify the following:

$$-(d/dx)P(Z \geq x, I = 1) = \frac{f(x)}{1 - F(x)}.$$ 

(2) Let $X_n, n = 1, 2, \cdots,$ and $X$ be real-valued random variables defined on the same probability space.

(a) Give an example in which $X_n \Rightarrow X$ in distribution, but $X_n \not\rightarrow X$ in probability.

(b) Suppose $X_n \Rightarrow X$ in distribution and $X = c$, where $c$ is a deterministic number. Show that $X_n \rightarrow c$ in probability.

(c) Give an example in which $X_n \Rightarrow X$ in distribution, but 

$$\lim_{n \to \infty} E[X_n] \neq E[X].$$ 

(d) Let $X_1, X_2, \ldots$ be i.i.d. with mean 1 and finite variance $\sigma^2$. Show that $(1/n) \sum_{i=1}^{n} X_iX_{i+1}$ converges in probability, and identify the limit.

(3) Let $X$ and $Y$ be bounded, real-valued random variables on a probability space $(\Omega, \mathcal{F}, P)$. Assume $\mathcal{G}$ is a sub-sigma field of $\mathcal{F}$. Prove that 

$$E[Y \mathbb{E}[X | \mathcal{G}]] = E[\mathbb{E}[Y | \mathcal{G}]]$$ 

using the definition of conditional expectation.

(4) Let $X_1, \cdots, X_n$ be a random sample from an exponential distribution with unknown mean $\theta$, i.e., $p(x) = \theta^{-1} \exp(-x/\theta)$, for $x > 0$ and $\theta > 0$.

(a) Find the MLE of the 75th percentile, say $q$, of the distribution.

(b) Determine whether or not this estimator is unbiased.
(c) Calculate the mean squared error (MSE) of the estimator in terms of $n$ and $\theta$. Does $MSE \to 0$ as $n \to \infty$?

(5) Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be two independent random samples from $N(\mu_1, \sigma^2_1)$ and $N(\mu_2, \sigma^2_2)$, respectively, where all parameters are unknown. Let $\overline{X}_1$ and $\overline{X}_2$ denote the sample means; $S^2_1$ and $S^2_2$ denote the sample variances of the two samples, respectively.

(a) Write down (without proof) the MLEs, $\hat{\mu}_1, \hat{\sigma}^2_1, \hat{\mu}_2, \hat{\sigma}^2_2$ for $\mu_1, \sigma^2_1, \mu_2, \sigma^2_2$, respectively.

(b) Derive the $\alpha$-level likelihood ratio test for $H_0 : \sigma^2_1 = \sigma^2_2$ against $H_1 : \sigma^2_1 \neq \sigma^2_2$. The resulting test should be expressed in terms of a well-known statistic. (You need not derive the usual MLEs, but you do need to justify the likelihood ratio test.)

(6) Let $X_1, \ldots, X_n$ be i.i.d. random variables from $\text{Binomial}(r, \theta)$, where $0 < \theta < 1$ and $r \geq 1$ is an integer.

(a) Justify that $T = \sum_{i=1}^{n} X_i$ is a complete and sufficient statistic for $\theta$.

(b) Write $q = \Pr[X_1 \leq 1]$ in terms of $\theta$, and define a random indicator $U$ which is an unbiased estimator for $q$.

(c) Use the properties of $T$ and the Rao-Blackwell Theorem to find the UMVU estimator of $q$. 