DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM – PROBABILITY
January 22, 2001

Work all problems. 60 points are needed to pass at the Master's level and 75 to pass at the Ph.D. level.

1. (20 pts) Two cards are drawn without replacement from a standard deck.

   (a) Assuming that the order in which the cards are drawn is important, describe an appropriate sample space for this situation.

   (b) Let A_1 be the event "first card is an ace," and A_2 be the event "second card is an ace." Use the *definition* of conditional probability to compute $P(A_2|A_1)$.

   (c) Consider now an arbitrary sample space Ω and two events C and B in Ω, with $P(C) > 0$. Suppose that $C_1, C_2, \ldots, C_k$ are disjoint sets, $P(C_i) > 0$, with $C = \cup_{i=1}^k C_i$. Further suppose that $P(B|C_i) = P(B|C_1)$ for $i = 2, \ldots, k$. Show that $P(B|C) = P(B|C_1)$.

   (d) Returning to the card experiment, let B_1 be the event "first card is ace of spaces." Compute $P(A_2|B_1)$. How would the answer change if the first card were the ace of diamonds.

2. (20 pts) Consider the following problems involving independent flips of fair coins.

   (a) Flip a coin until the last two flips are HT and let N be the number of flips required. What is the pdf of N?

   (b) Begin by flipping k coins simultaneously. After the first simultaneous flip remove all the coins which showed heads and flip the remaining coins simultaneously. Continue this process until no coins are left and let T be the number of the last simultaneous flip. Compute the cdf and pdf of T.

3. (20 pts) Let X_1, X_2, and X_3 be independent random variables such that X_i has a Gamma distribution with paramters α_i, λ_i. That is, X_i has density

$$f_i(x) = 1_{(0,\infty)}(x)\frac{\lambda^{\alpha_i}}{\Gamma(\alpha_i)}x^{\alpha_i-1}e^{-\lambda_i x}$$

   where $\alpha_i > 0$ and $\lambda_i > 0$. Let

$$Y_1 = \frac{X_1}{X_1+X_2+X_3}$$
$$Y_2 = \frac{X_2}{X_1+X_2+X_3}$$
$$Y_3 = X_1+X_2+X_3$$

   (a) Find the joint density of Y_1, Y_2, Y_3.

   (b) Find the marginal joint density of Y_1, Y_2.

   (c) Show that (Y_1, Y_2) and Y_3 are independent.

4. (15 pts) From an urn containing 10 balls numbered 0 through 9, n balls are drawn with replacement. Let $X_i = 1$ if the ith draw yields the ball numbered 0, and $X_i = 0$ otherwise, $i = 1, \ldots, n$.

   (a) What does the weak law of large numbers tell you about the occurrence of 0's in the n drawings?

(b) Use the central limit theorem to find an approximate probability that, among the n balls thus chosen, the ball numbered 0 will appear between $(n - 3\sqrt{n})/10$ and $(n + 3\sqrt{n})/10$ if $n = 100$.

5. (25 pts)

(a) A random variable has a $\chi^2(\nu)$ distribution if its pdf is

$$f(x) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

with MGF

$$M_X(t) = \left(\frac{1}{1-2t}\right)^{\nu/2}$$

Let $X_1, X_2, \ldots$ be independent with $X_i \sim \chi^2(\nu_i)$ What is the pdf of $Y = \sum_{i=1}^{n} X_i$?

(b) A random variable X has moment generating function

$$M_X(t) = \frac{1}{2}e^t + \frac{1}{4}e^{2t} + \frac{1}{4}e^{5t}$$

What is the distribution of X? Why?