ADVANCED EXAM: TOPOLOGY, AUGUST 2021

Answer all seven questions. Justify your answers.
Passing standard: 70% with four questions essentially complete.

1. Show that a continuous map $S^1 \to \mathbb{R}$ can be neither injective nor surjective.

2. Prove that a metric space has a countable dense subset if and only if it has a countable basis for its topology.

3. Let $q\colon M \to N$ be a finite (connected) covering. Show that

(a) if N is compact, then M is compact,
(b) if N is Hausdorff, then M is Hausdorff,
(c) if N is locally Euclidean, then M is locally Euclidean.

4. Let X be the quotient space obtained from a disjoint union of two 2–spheres by identifying their North Poles and then their South Poles. (Picture this space.)

(a) Describe carefully a CW structure on X and then calculate $\pi_1(X)$ *using this CW structure*. Identify the group you get.
(b) Does X have any non-normal (non-regular) covers? Justify your answer.

5. Calculate the integral homology and cohomology *groups* of $X = (S^1 \times \mathbb{CP}^2) \# (S^3 \times \mathbb{RP}^2)$.

6. Prove that $\mathbb{RP}^n$ does not retract onto $\mathbb{RP}^k$ for any $n > k > 0$.

7. Let Y be a compact, connected and non-orientable n–manifold without boundary.

(a) Prove that if $n = 3$, then $\pi_1(Y)$ is necessarily infinite.
(b) Show that in any other dimension $n \geq 2$, there is such Y with finite $\pi_1(Y)$.