Analysis Qualifying Examination

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Friday, August 27th, 2020

This exam consists of eight equally weighted problems (ten points each): a passing grade is 65% (52/80), including at least five "essentially correct" problems ($\approx 7.5/10$).

Clearly show your work, explicitly stating or naming results that you use; justify use of named theorems by verifying necessary conditions.

Please work legibly and clearly label each page/file of your exam with your name.

1. Suppose $f_1, f_2, \ldots$, and f are in $L^1_{loc}(U)$ for some open set $U \subseteq \mathbb{R}^n$ (that is, they are integrable on any compact subset of U). Which of the following conditions imply that $f_n \to f$ in $\mathcal{D}'(U)$ (that is, in the sense of distributions)? Justify your answer.
   (a) $f_n, f \in L^p(U)$ for some $p \in (1, \infty)$ and $f_n \to f$ weakly in L^p.
   (b) There is $g \in L^1_{loc}(U)$ such that $|f_n| \leq g$ for all n and $f_n \to f$ almost everywhere.
   (c) $f_n \to f$ pointwise.

2. Let μ, ν, ν_1, ν_2 be measures on $(\Omega, \mathcal{F})$. Prove the following assertions.
   (a) If $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$ then $\nu_1 + \nu_2 \perp \mu$.
   (b) If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$ then $\nu_1 + \nu_2 \ll \mu$.
   (c) If $\nu \perp \mu$ and $\nu \ll \mu$ then $\nu = 0$.

3. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.

(a) Suppose $(x_n)_{n=1}^{\infty}$ is a sequence of pairwise orthogonal vectors. Prove that the following are equivalent:

   i. $\sum_{n=1}^{\infty} x_n$ converges in the norm topology of H.

   ii. $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$.

   iii. $\sum_{n=1}^{\infty} \langle x_n, y\rangle$ converges for every $y \in H$.

(b) Suppose now that we do not assume $(x_n)_{n=1}^{\infty}$ to be pairwise orthogonal. Are i. and iii. above equivalent? Prove or give a counter-example.

4. Suppose $f \in C_c(0,\infty)$ with $f \geq 0$, and let

$$F(x) = \frac{1}{x}\int_0^x f(y)dy, \qquad x \in (0,\infty).$$

   (a) For any $p \in (1,\infty)$, prove that $F \in L^p(0,\infty)$ and that in fact

$$\|F\|_{L^p(0,\infty)} \leq \frac{p}{p-1}\|f\|_{L^p(0,\infty)}.$$

   [Hint: Write $F^p(x) = F^p(x)\frac{d}{dx}x$, integrate by parts, and note that $xF' = f - F$. Use Hölder's inequality.]

   (b) If $f \neq 0$, prove that $F \notin L^1(0,\infty)$.

5. Let $\mathcal{H}$ be a Hilbert space. A linear map $P : \mathcal{H} \to \mathcal{H}$ is called an *idempotent* if $P^2 = P$, and a self-adjoint idempotent is called a *projection.* Denote the range and kernel (or nullspace) of P by $\mathcal{R}$ and $\mathcal{N}$, respectively.

   (a) Show that P is an idempotent or projection if and only if $I - P$ is too;

   (b) Fully characterize the eigenvalues and eigenspaces of an idempotent P in terms of $\mathcal{R}$ and $\mathcal{N}$;

   (c) Show that an idempotent P is a projection if and only if $\mathcal{R} \perp \mathcal{N}$.

6. Show that if f is uniformly continuous and integrable on all of $\mathbb{R}^d$, then $f(x) \to 0$ as $|x| \to \infty$. On the other hand, find an example of a continuous integrable $f \geq 0$ such that $\limsup_{|x|\to\infty} f(x) = \infty$, so uniform continuity is *necessary* to conclude $f(x) \to 0$ as $|x| \to \infty$.

7. Recall Vitali's nonmeasurable set $\mathcal{V} \subset [0,1]$: (declare $a \sim b$ iff $a - b \in \mathbb{Q}$; then $\mathcal{V}$ consists of exactly one representative from each equivalence class), and let $\mathcal{W} = [0,1]\backslash\mathcal{V}$.

   (a) Show that $m^*(\mathcal{W}) \geq 1$. [Hint: Recall that any measurable subset of $\mathcal{V}$ has measure zero.]

   (b) Conclude that

   $$m^*(\mathcal{V} \cap [0,1]) + m^*(\mathcal{V}^c \cap [0,1]) > m^*([0,1]),$$

   so $\mathcal{V}$ fails Caratheodory's definition of measurability.

8. The following integral equation for $f : [-a,a] \to \mathbb{R}$ arises in a model for the motion of gas particles on a line:

   $$f(x) = 1 + \frac{1}{\pi} \int_{-a}^{a} \frac{1}{1 + (x-y)^2} f(y)\, dy, \quad \text{for} \quad -a \leq x \leq a.$$

   For any fixed $a \in (0, \infty)$, show that this equation has a unique, bounded and continuous solution. [Hint: Use the Contraction Mapping Theorem.]