This exam consists of eight equally weighted problems (ten points each): a passing grade is 65% (52/80), including at least five “essentially correct” problems (≈ 7.5/10).

Clearly show your work, explicitly stating or naming results that you use; justify use of named theorems by verifying necessary conditions.

Please work legibly and clearly label each page/file of your exam with your name.

1. Suppose \( f_1, f_2, \ldots, \) and \( f \) are in \( L^1_{\text{loc}}(U) \) for some open set \( U \subseteq \mathbb{R}^n \) (that is, they are integrable on any compact subset of \( U \)). Which of the following conditions imply that \( f_n \rightarrow f \) in \( D'(U) \) (that is, in the sense of distributions)? Justify your answer.

   (a) \( f_n, f \in L^p(U) \) for some \( p \in (1, \infty) \) and \( f_n \rightarrow f \) weakly in \( L^p \).

   (b) There is \( g \in L^1_{\text{loc}}(U) \) such that \( |f_n| \leq g \) for all \( n \) and \( f_n \rightarrow f \) almost everywhere.

   (c) \( f_n \rightarrow f \) pointwise.

2. Let \( \mu, \nu, \nu_1, \nu_2 \) be measures on \((\Omega, \mathcal{F})\). Prove the following assertions.

   (a) If \( \nu_1 \perp \mu \) and \( \nu_2 \perp \mu \) then \( \nu_1 + \nu_2 \perp \mu \).

   (b) If \( \nu_1 \ll \mu \) and \( \nu_2 \ll \mu \) then \( \nu_1 + \nu_2 \ll \mu \).

   (c) If \( \nu \perp \mu \) and \( \nu \ll \mu \) then \( \nu = 0 \).

3. Let \( H \) be a Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \).
(a) Suppose \((x_n)_{n=1}^{\infty}\) is a sequence of pairwise orthogonal vectors. Prove that the following are equivalent:
   i. \(\sum_{n=1}^{\infty} x_n\) converges in the norm topology of \(H\).
   ii. \(\sum_{n=1}^{\infty} \|x_n\|^2 < \infty\).
   iii. \(\sum_{n=1}^{\infty} \langle x_n, y \rangle\) converges for every \(y \in H\).
(b) Suppose now that we do not assume \((x_n)_{n=1}^{\infty}\) to be pairwise orthogonal. Are i. and iii. above equivalent? Prove or give a counter-example.

4. Suppose \(f \in C_c(0, \infty)\) with \(f \geq 0\), and let
   \[ F(x) = \frac{1}{x} \int_0^x f(y) dy, \quad x \in (0, \infty). \]
   (a) For any \(p \in (1, \infty)\), prove that \(F \in L^p(0, \infty)\) and that in fact
   \[ \|F\|_{L^p(0, \infty)} \leq \frac{p}{p-1} \|f\|_{L^p(0, \infty)}. \]
   [Hint: Write \(F^p(x) = F^p(x) \frac{d}{dx} x\), integrate by parts, and note that \(xF' = f - F\). Use Hölder’s inequality.]
   (b) If \(f \neq 0\), prove that \(F \notin L^1(0, \infty)\).

5. Let \(\mathcal{H}\) be a Hilbert space. A linear map \(P : \mathcal{H} \to \mathcal{H}\) is called an idempotent if \(P^2 = P\), and a self-adjoint idempotent is called a projection. Denote the range and kernel (or nullspace) of \(P\) by \(\mathcal{R}\) and \(\mathcal{N}\), respectively.
   (a) Show that \(P\) is an idempotent or projection if and only if \(I - P\) is too;
   (b) Fully characterize the eigenvalues and eigenspaces of an idempotent \(P\) in terms of \(\mathcal{R}\) and \(\mathcal{N}\);
   (c) Show that an idempotent \(P\) is a projection if and only if \(\mathcal{R} \perp \mathcal{N}\).

6. Show that if \(f\) is uniformly continuous and integrable on all of \(\mathbb{R}^d\), then \(f(x) \to 0\) as \(|x| \to \infty\). On the other hand, find an example of a continuous integrable \(f \geq 0\) such that \(\lim \sup_{|x| \to \infty} f(x) = \infty\), so uniform continuity is necessary to conclude \(f(x) \to 0\) as \(|x| \to \infty\).
7. Recall Vitali’s nonmeasurable set $\mathcal{V} \subset [0,1]$: (declare $a \sim b$ iff $a - b \in \mathbb{Q}$; then $\mathcal{V}$ consists of exactly one representative from each equivalence class), and let $\mathcal{W} = [0,1] \setminus \mathcal{V}$.

(a) Show that $m^*(\mathcal{W}) \geq 1$. [Hint: Recall that any measurable subset of $\mathcal{V}$ has measure zero.]

(b) Conclude that

$$m^*(\mathcal{V} \cap [0,1]) + m^*(\mathcal{V}^c \cap [0,1]) > m^*([0,1]),$$

so $\mathcal{V}$ fails Caratheodory’s definition of measurability.

8. The following integral equation for $f : [-a,a] \to \mathbb{R}$ arises in a model for the motion of gas particles on a line:

$$f(x) = 1 + \frac{1}{\pi} \int_{-a}^{a} \frac{1}{1 + (x - y)^2} f(y) \, dy, \quad \text{for} \quad -a \leq x \leq a.$$ 

For any fixed $a \in (0,\infty)$, show that this equation has a unique, bounded and continuous solution. [Hint: Use the Contraction Mapping Theorem.]