1. Let \( \{X_n\}_{n \geq 1} \) be a sequence of real-valued random variables with distribution function \( F_n(x) \) for each \( n \) and \( X \) be another real-valued random variable with distribution function \( F(x) \).

(a) (3 points) State the definition of almost sure convergence (denoted as \( X_n \overset{a.s.}{\rightarrow} X \)).

(b) (3 points) State the definition of convergence in \( a^t \)th mean (denoted as \( X_n \overset{a^t}{\rightarrow} X \)).

(c) (3 points) State the definition of convergence in probability (denoted as \( X_n \overset{P}{\rightarrow} X \)).

(d) (3 points) State the definition of convergence in distribution (denoted as \( X_n \overset{d}{\rightarrow} X \)).

(e) (5 points) Prove that for fixed \( a > 0 \), \( X_n \overset{a}{\rightarrow} X \) implies \( X_n \overset{P}{\rightarrow} X \).

(f) (6 points) Show that \( \bar{X}_n = \mu + O_p(1/\sqrt{n}) \) as \( n \to \infty \) where \( \bar{X}_n \) is the sample mean of \( n \) independent and identically distributed random variables with mean \( \mu \) and finite variance \( \sigma^2 \). Note that the definition of \( O_p \) is as follows: the sequence of real numbers \( \{c_n\}_{n \geq 1} \) is asymptotically equivalent to the sequence \( \{d_n\}_{n \geq 1} \), written as \( c_n \sim d_n \) if \( (c_n / d_n) \to 1 \) as \( n \to \infty \).

2. Suppose \( Y_1, Y_2, \ldots, \) is a simple random sample from an exponential distribution with density \( g(y) = \theta \exp(-\theta y) \) where \( \theta > 0 \) and \( y > 0 \). Consider the estimator of \( h(\theta) = 1/\theta \), \( \hat{h}_n = \sum_{i=1}^n Y_i / (n+2) \).

(a) (4 points) Compute the bias of \( \hat{h}_n \), denoted as \( B(\hat{h}_n) \).

(b) (4 points) Compute the variance of \( \hat{h}_n \), denoted as \( Var(\hat{h}_n) \).

(c) (4 points) Show that \( B(\hat{h}_n) \sim k_1 Var(\hat{h}_n) \sim k_2 (1/n) \) as \( n \to \infty \) for some constants \( k_1 \) and \( k_2 \) depending on \( \theta \). Note that the definition of \( \sim \) is as follows: the sequence of real numbers \( \{c_n\}_{n \geq 1} \) is asymptotically equivalent to the sequence \( \{d_n\}_{n \geq 1} \), written as \( c_n \sim d_n \) if \( (c_n / d_n) \to 1 \) as \( n \to \infty \).

3. Suppose \( X_1, X_2, \ldots, \) are independent and identically distributed with mean \( \mu \) and variance \( \sigma^2 \). Let \( Y_i = \bar{X}_i = (\sum_{j=1}^i X_j) / i \).

(a) (3 points) Are \( Y_1, Y_2, \ldots \) independent? Verify your answer.

(b) (7 points) Show that \( \bar{Y}_n = (\sum_{i=1}^n Y_i) / n \) is a consistent estimator of \( \mu \).

[Hint] Use \( Var(\bar{Y}_n) = \sigma^2 / n^2 (2n - \sum_{i=1}^n 1/i) \).
(c) (8 points) Compute the limit of the relative efficiency of $\bar{Y}_n$ to $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$, defined to be $\frac{\text{Var}(\bar{X}_n)}{\text{Var}(\bar{Y}_n)}$ as $n \to \infty$.

4. Answer the following questions.

(a) (10 points) Suppose that $X$ is a random variable with $E(X) = 0$ and $\text{Var}(X) = \sigma^2 < \infty$. Let $Z_n$ denote the random variable $X^2 I[|X| \geq \sigma \sqrt{n}]$. Prove that $E(Z_n) \to 0$ as $n \to \infty$.

(b) (15 points) Suppose that $X_1, X_2, \ldots$, are independent and identically distributed with $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2 < \infty$. Let $k_{n1}, \ldots, k_{nn}$ be constants satisfying

$$\max_{1 \leq i \leq n} k_{ni}^2 \sum_{j=1}^{n} k_{nj}^2 \to 0 \quad \text{as} \quad n \to \infty.$$ 

Let $T_n = \sum_{i=1}^{n} k_{ni} X_i$. Prove that

$$\frac{T_n - E(T_n)}{\sqrt{\text{Var}(T_n)}} \overset{d}{\to} N(0, 1).$$

[Hint] Use Lindeberg-Feller Central Limit Theorem by checking the Lindeberg condition.

5. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed with the unknown distribution function $P(X_i \leq x) = F(x - \mu)$ for the mean parameter $\mu$ and the variance parameter $\sigma^2$ where $f(x) = F'(x)$ exists and is symmetric about 0. Note that $\sigma^2$ is not a function of $\mu$.
We want to test $H_0 : \mu = 0$ vs. $H_1 : \mu > 0$. Consider the statistic $\bar{X}_n = \sum_{i=1}^{n} X_i/n$.

(a) (3 points) Prove that under $H_0 : \mu = 0$,

$$\sqrt{n} \bar{X}_n / \sigma \overset{d}{\to} Z,$$

where $Z$ is the standard normal distribution with mean 0 and variance 1.

(b) (3 points) Assume that one rejects $H_0 : \mu = 0$ whenever

$$\bar{X}_n > u_{\alpha} \sigma / \sqrt{n},$$

where $u_{\alpha}$ is the $1 - \alpha$ quantile of the standard normal distribution. Show that this test has asymptotic level $\alpha$ using (a).

(c) (10 points) Show that the following asymptotic result holds for the alternatives $\{\mu_n\}$ satisfying $\mu_n > 0$ for all $n$:

$$\sqrt{n} \frac{(\bar{X}_n - \mu_n)}{\sigma} \overset{d}{\to} Z,$$

where $\mu_n$ means that the mean parameter depends on $n$.

(d) (6 points) Suppose that $\mu_n > 0$ for all $n$ and $\sqrt{n} \mu_n \to \delta > 0$. Let $\text{Power}_n(\mu_n)$ be the power of this test against the alternative $\mu_n$. Show that

$$\text{Power}_n(\mu_n) \to \Phi \left( \frac{\delta}{\sigma} - u_{\alpha} \right) \quad \text{as} \quad n \to \infty$$

where $\Phi(x)$ denotes the cumulative distribution function of the standard normal distribution.