1. Let $X_1, \ldots, X_n$ be a random sample drawn from a Poisson distribution with mean $\lambda$,

$$f(x \mid \lambda) = e^{-\lambda} \frac{\lambda^x}{x!},$$

where $\lambda > 0$ is an unknown parameter. Consider two estimators for $\lambda$, $T_1 = \frac{1}{n} \sum_{i=1}^{n} X_i$ and $T_2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$.

(a) Are $T_1$ and $T_2$ unbiased estimators for $\lambda$? Justify your answer.

(b) Which estimator is more efficient for $\lambda$ between $T_1$ and $T_2$? Justify your answer.

(c) Calculate the Cramér-Rao lower bound for unbiased estimators of $\lambda^2$.

(d) Suppose that $\lambda$ has an exponential prior distribution with mean $\theta > 0$,

$$f(\lambda \mid \theta) = \frac{1}{\theta} e^{-\lambda/\theta}.$$

Derive the posterior distribution of $\lambda$.

(e) Compute the posterior mean of $\lambda$ and show that the posterior mean of $\lambda$ is consistent for $\lambda$ as $n \to \infty$.

2. Let $X_1, \ldots, X_n$ be a random sample drawn from the following probability density function

$$f(x \mid \lambda) = \frac{1}{\lambda} x^{\lambda - 1}$$

where $0 \leq x \leq 1$ and $\theta > 0$.

(a) Find the distribution of $-2 \log X_i$ where $i = 1, \ldots, n$.

(b) Show that $S = -2 \sum_{i=1}^{n} \log(X_i)$ is a minimal sufficient statistic for $\lambda$.

(c) Obtain a two-sided 95% confidence interval for $\lambda$. [Hint] Consider the pivotal quantity using $S = -2 \sum_{i=1}^{n} \log(X_i)$.

(d) Compute the length of the confidence interval obtained in (c) and check if its expectation (i.e., the expected length of the confidence interval) converges to zero as $n \to \infty$.
3. For $\theta > 0$, let $Y_1, \ldots, Y_n$ be independent and identically distributed with probability density function given by

$$f_Y(y) = \begin{cases} \frac{3y^2}{\theta^3}, & 0 \leq y \leq \theta, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Show that $Y_{(n)} = \max(Y_1, \ldots, Y_n)$ is sufficient for $\theta$.
(b) Show that the pdf of $Y_{(n)}$ is given by

$$f_{Y_{(n)}}(y) = \begin{cases} \frac{3ny^{3n-1}}{\theta^{3n}}, & 0 \leq y \leq \theta, \\ 0, & \text{elsewhere.} \end{cases}$$

(c) Find the MLE for $\theta$.
(d) Construct a pivotal quantity for this model by a simple transformation of the MLE you obtained.
(e) Use the pivotal quantity you obtained in (d) to give a $100(1 - \alpha)\%$ confidence interval for $\theta$.

4. For $\theta \in (0, 1)$, let $Y_1, \ldots, Y_n$ be independent and identically distributed with probability mass function given by

$$\Pr(Y_j = y_j) = \begin{cases} \theta^2, & y_j = 1 \\ 2\theta(1 - \theta), & y_j = 2 \\ (1 - \theta)^2, & y_j = 3. \end{cases}$$

Let $N_i$ denote the number of observations equal to $i$ for $i = 1, 2, 3$.

(a) Give an expression for the likelihood function $L(\theta)$ in terms of $N_1, N_2,$ and $N_3$.
(b) Find the form of the rejection region for the most powerful test of $H_0 : \theta = \theta_0$ versus $H_a : \theta = \theta_a$, where $\theta_0 < \theta_a$. Your test should reject for certain values of $2N_1 + N_2$.
(c) Is a level $\alpha$ test based on your rejection region obtained in (b) uniformly most powerful for testing $H_0 : \theta = \theta_0$ versus $H_a : \theta > \theta_0$? Justify your answer. Note that a uniformly most powerful test is a hypothesis test which has the greatest power among all possible tests of a given level $\alpha$. 