UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Basic Exam - Statistics
Tuesday, August 18, 2020

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level. Each answer is worth approximately the same number of points.

1. Let $X_1, \ldots, X_n$ be a random sample drawn from a Poisson distribution with mean λ,

$$f(x \mid \lambda) = e^{-\lambda}\frac{\lambda^x}{x!},$$

where $\lambda > 0$ is an unknown parameter. Consider two estimators for λ, $T_1 = \frac{1}{n}\sum_{i=1}^n X_i$ and $T_2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$.

   (a) Are T_1 and T_2 unbiased estimators for λ? Justify your answer.

   (b) Which estimator is more efficient for λ between T_1 and T_2? Justify your answer.

   (c) Calculate the Cramér-Rao lower bound for unbiased estimators of λ^2.

   (d) Suppose that λ has an exponential prior distribution with mean $\theta > 0$,

$$f(\lambda \mid \theta) = \frac{1}{\theta}e^{-\lambda/\theta}.$$

   Derive the posterior distribution of λ.

   (e) Compute the posterior mean of λ and show that the posterior mean of λ is consistent for λ as $n \to \infty$.

2. Let $X_1, \ldots, X_n$ be a random sample drawn from the following probability density function

$$f(x \mid \lambda) = \frac{1}{\lambda}x^{\frac{1}{\lambda}-1}$$

where $0 \leq x \leq 1$ and $\theta > 0$.

   (a) Find the distribution of $-2\log X_i$ where $i = 1, \ldots, n$.

   (b) Show that $S = -2\sum_{i=1}^n \log(X_i)$ is a minimal sufficient statistic for λ.

   (c) Obtain a two-sided 95% confidence interval for λ. [Hint] Consider the pivotal quantity using $S = -2\sum_{i=1}^n \log(X_i)$.

   (d) Compute the length of the confidence interval obtained in (c) and check if its expectation (i.e., the expected length of the confidence interval) converges to zero as $n \to \infty$.

3. For $\theta > 0$, let $Y_1, \ldots, Y_n$ be independent and identically distributed with probability density function given by

$$f_Y(y) = \begin{cases} \frac{3y^2}{\theta^3}, & 0 \le y \le \theta, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) Show that $Y_{(n)} = \max(Y_1, \ldots, Y_n)$ is sufficient for θ.

   (b) Show that the pdf of $Y_{(n)}$ is given by

$$f_{Y_{(n)}}(y) = \begin{cases} \frac{3ny^{3n-1}}{\theta^{3n}}, & 0 \le y \le \theta, \\ 0, & \text{elsewhere.} \end{cases}$$

   (c) Find the MLE for θ.

   (d) Construct a pivotal quantity for this model by a simple transformation of the MLE you obtained.

   (e) Use the pivotal quantity you obtained in (d) to give a $100(1-\alpha)\%$ confidence interval for θ.

4. For $\theta \in (0, 1)$, let $Y_1, \ldots, Y_n$ be independent and identically distributed with probability mass function given by

$$\Pr(Y_j = y_j) = \begin{cases} \theta^2, & y_j = 1 \\ 2\theta(1-\theta), & y_j = 2 \\ (1-\theta)^2, & y_j = 3. \end{cases}$$

   Let N_i denote the number of observations equal to i for $i = 1, 2, 3$.

   (a) Give an expression for the likelihood function $L(\theta)$ in terms of N_1, N_2, and N_3.

   (b) Find the form of the rejection region for the most powerful test of $H_0 : \theta = \theta_0$ versus $H_a : \theta = \theta_a$, where $\theta_0 < \theta_a$. Your test should reject for certain values of $2N_1 + N_2$.

   (c) Is a level α test based on your rejection region obtained in (b) uniformly most powerful for testing $H_0 : \theta = \theta_0$ versus $H_a : \theta > \theta_0$? Justify your answer. Note that a uniformly most powerful test is a hypothesis test which has the greatest power among all possible tests of a given level α.