Advanced Calculus/Linear algebra basic exam

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Instructions: Do 7 of the 8 problems. Show your work. The passing standards are:

- Master’s level: 60% with three questions essentially, complete (including one question from each part);
- Ph.D. level: 75% with two questions from each part essentially complete.

Calculus

1. Answer each of the following and explain your work.

   (a) \[ \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2}{\sqrt{1 + \frac{2i}{n}}} \]

   (b) \[ \lim_{h \to 0} \frac{1}{h} \int_{2}^{2+h} \frac{t}{\sqrt{1 + t^3}} dt. \]

   (c) \[ \lim_{x \to 1} \frac{\log(x^3)}{x^2 - 1} \]

   (d) Evaluate \[ \int_{0}^{\sqrt{5/2}} \frac{1}{\sqrt{25 - x^2}} dx. \]

2. Consider the function \( f(x, y) = xy - x^2 y^3 \). Find the maximimum and minimum values and where they are attained of \( f(x, y) \) when \((x, y)\) is in the square \( 0 \leq x \leq 1, 0 \leq y \leq 1 \). (Hint: \( \sqrt{3} \approx 0.73 \))

3. In the \( xy \)-plane, the disk \( \{x^2 + y^2 \leq 2x\} \) is cut into two pieces by the line \( y = x \). Let \( D \) be the larger piece.

   (a) Sketch \( D \) including an accurate description of the center and radius of the given disk.

   (b) Describe \( D \) in polar coordinates.

   (c) Find the volume of the solid below the surface \( \{z = \sqrt{x^2 + y^2}\} \) and above \( D \) in \( \mathbb{R}^3 \).

4. Let \( C \) be the curve obtained by intersecting the cylinder \( \{x^2 + y^2 = 1\} \) and the surface \( \{z = y^2\} \) oriented in the counterclockwise direction viewed from the positive \( z \)-axis. Let \( \mathbf{F} = \langle x^2 - y, y^2 + x, 1 \rangle \). Calculate the integral \( \oint_C \mathbf{F} \cdot d\mathbf{r} \) by

   (a) direct evaluation and

   (b) another method.
Linear Algebra

1. Consider the following two vectors

\[ v_1 = (1, 2, 2) \text{ and } v_2 = (-1, 0, 2). \]

(a) Give an orthonormal basis for the subspace \( V \subset \mathbb{R}^3 \) that they span.

(b) Find a unit vector orthogonal to \( V \).

2. View the complex numbers \( \mathbb{C} \) as a two-dimensional real vector space with basis \( \{1, i\} \).

(a) Given the complex number \( \alpha = a + ib \), express multiplication by \( \alpha \) as a real-valued matrix \( M_\alpha \).

(b) Does \( M_\alpha \) preserve angles between vectors?

Recall that for an \( n \times n \) matrix \( A \), then

\[ p_A(t) = \det(tI - A) = t^n + c_1(A)t^{n-1} + \cdots + c_n(A). \]

is its \textit{characteristic polynomial}.

3. Let

\[ A = \begin{pmatrix}
  2 & 0 & 1 & -3 \\
  0 & 2 & 10 & 4 \\
  0 & 0 & 2 & 0 \\
  0 & 0 & 0 & 3
\end{pmatrix}. \]

(a) Find the roots of its characteristic polynomial.

(b) Is this matrix diagonalizable? Find a maximal set of linearly independent eigenvectors of \( A \).

4. Let \( A \) be an arbitrary \( n \times n \) matrix.

(a) What is the constant term \( c_n(A) \) of its characteristic polynomial \( p_A(t) \)?

(b) The Cayley-Hamilton theorem says that such a square matrix \( A \) satisfies its own characteristic polynomial: \( p_A(A) = 0 \). Assuming the constant term of \( p_A \) is nonzero, express the inverse \( A^{-1} \) as a polynomial in \( A \). (The coefficients depend in a simple way on the coefficients \( \{c_k(A)\} \).

5. Given a matrix \( M \), let ker\( (M) \) denote the kernel/nullspace of the matrix and let im\( (M) \) denote the image/range.

(a) Let \( M \) be a \( 15 \times 15 \) matrix. Can ker\( (M) = \text{im}(M) \)?

(b) If the rank of \( M^2 \) equals the rank of \( M \), what do you know about \( \text{im}(M) \cap \ker(M) \)?