Advanced Calculus/Linear algebra basic exam

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Instructions: Do 7 of the 8 problems. Show your work. The passing standards are:

- Master's level: 60% with three questions essentially, complete (including one question from each part);
- Ph.D. level: 75% with two questions from each part essentially complete.

Calculus

1. Answer each of the following and explain your work.

   (a) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{1+\frac{2i}{n}}}\frac{2}{n}$

   (b) $\displaystyle\lim_{h\to 0}\frac{1}{h}\int_{2}^{2+h}\frac{t}{\sqrt{1+t^3}}dt.$

   (c) $\displaystyle\lim_{x\to 1}\frac{\log(x^3)}{x^2-1}$

   (d) Evaluate $\displaystyle\int_{0}^{5/2}\frac{1}{\sqrt{25-x^2}}dx.$

2. Consider the function $f(x,y)=xy-x^2y^3$. Find the maximium and minimum values and where they are attained of $f(x,y)$ when (x,y) is in the square $0\le x\le 1, 0\le y\le 1$. (*Hint:* $\sqrt{3}\approx 0.73$)

3. In the xy-plane, the disk $\{x^2+y^2\le 2x\}$ is cut into two pieces by the line $y=x$. Let D be the larger piece.

   (a) Sketch D including an accurate description of the center and radius of the given disk.

   (b) Describe D in polar coordinates.

   (c) Find the volume of the solid below the surface $\{z=\sqrt{x^2+y^2}\}$ and above D in $\mathbb{R}^3$.

4. Let C be the curve obtained by intersecting the cylinder $\{x^2+y^2=1\}$ and the surface $\{z=y^2\}$ oriented in the counterclockwise direction viewed from the positive z-axis. Let $\mathbf{F}=\langle x^2-y, y^2+x, 1\rangle$. Calculate the integral $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r}$ by

   (a) direct evaluation and

   (b) another method.

Linear Algebra

1. Consider the following two vectors

$$v_1 = (1, 2, 2) \text{ and } v_2 = (-1, 0, 2).$$

   (a) Give an orthonormal basis for the subspace $V \subset \mathbb{R}^3$ that they span.

   (b) Find a unit vector orthogonal to V.

2. View the complex numbers $\mathbb{C}$ as a two-dimensional real vector space with basis $\{1, i\}$.

   (a) Given the complex number $\alpha = a + ib$, express multiplication by α as a real-valued matrix M_α.

   (b) Does M_α preserve angles between vectors?

   Recall that for an $n \times n$ matrix A, then

$$p_A(t) = \det(tI - A) = t^n + c_1(A)t^{n-1} + \cdots + c_n(A).$$

   is its *characteristic polynomial*.

3. Let

$$A = \begin{pmatrix} 2 & 0 & 1 & -3 \\ 0 & 2 & 10 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

   (a) Find the roots of its characteristic polynomial.

   (b) Is this matrix diagonalizable? Find a maximal set of linearly independent eigenvectors of A.

4. Let A be an arbitrary $n \times n$ matrix.

   (a) What is the constant term $c_n(A)$ of its characteristic polynomial $p_A(t)$?

   (b) The Cayley-Hamilton theorem says that such a square matrix A satisfies its own characteristic polynomial: $p_A(A) = 0$. Assuming the constant term of p_A is nonzero, express the inverse A^{-1} as a polynomial in A. (The coefficients depend in a simple way on the coefficients $\{c_k(A)\}$.)

5. Given a matrix M, let $\ker(M)$ denote the kernel/nullspace of the matrix and let $\operatorname{im}(M)$ denote the image/range.

   (a) Let M be a 15×15 matrix. Can $\ker(M) = \operatorname{im}(M)$?

   (b) If the rank of M^2 equals the rank of M, what do you know about $\operatorname{im}(M) \cap \ker(M)$?