ADVANCED EXAM: TOPOLOGY, SUMMER 2020

Answer all seven questions. Justify your answers.
Passing standard: 70% with four questions essentially complete.

1. Let $F\colon X \to Y$ be a continuous bijection, where X is compact and Y is Hausdorff. Show that F is homeomorphism.

2. Let CX be the cone over the topological space X, i.e. $CX = ([0,1] \times X)/\sim$, where $0 \times x \sim 0 \times x'$ for every $x, x' \in X$. Prove that

(a) CX is path-connected
(b) CX is compact if X is
(c) CX is locally connected if and only if X is.

3. Find all 2 and 3-sheeted covers of $S^1 \vee S^1$ up to isomorphism of covering spaces without base point.

4. Show that any map $S^2 \to S^1$ is null homotopic. Then show, by an example, that the same does not hold for $T^2 \to S^1$.

5. $X \subset \mathbb{R}^3$ is the union of the 3 coordinate lines through the origin O. Let $Z = \mathbb{R}^3 \setminus X$.

(a) Compute $\pi_1(Z)$.
(b) Compute $H_i(Z, Z \setminus O)$ for all i.

6. Let X be the quotient $(S^1 \times I)/\sim$ by the following equivalence relation $\sim$:

- $(z, 0) \sim (w, 0)$ if and only if $z^4 = w^4$
- $(z, 1) \sim (w, 1)$ if and only if $z^{10} = w^{10}$
- in all other cases, $(z, s) \sim (w, t)$ only when $z = w$ and $s = t$.

Compute $H_i(X; G)$ and $H^i(X; G)$, for $G = \mathbb{Z}$ and $\mathbb{Z}_2$, respectively.

7. Prove that for any compact connected 3–manifold M, the Euler characteristic $\chi(M) = 0$. Recall that $\chi(M) = \sum_i (-1)^i \operatorname{rank}(H_i(M; G))$, where G can be taken as $\mathbb{Z}$ or any field.