Analysis Qualifying Examination

Department of Mathematics and Statistics
University of Massachusetts

Friday, August 21st, 2020

This exam consists of eight equally weighted problems (ten points each): a passing grade is 65% (52/80), including at least five "essentially correct" problems ($\approx 7.5/10$).

Clearly show your work, explicitly stating or naming results that you use; justify use of named theorems by verifying necessary conditions.

Please work legibly and clearly label each page/file of your exam with your name.

1. Let $\{q_n \mid n \geq 1\}$ be a countable dense set in $[0,1]$, and let (a_n) be a sequence of positive numbers satisfying $\sum_n a_n = a < \infty$. Consider the series

$$b(x) = \sum_n \frac{a_n^2}{|x - q_n|}.$$

   (a) Show that $b(x)$ is unbounded on any open subinterval of $[0,1]$.

   (b) For any $\beta > 0$, set

$$E_\beta = \left\{x \in [0,1] \;\middle|\; |x - q_n| \geq \beta\, a_n \text{ for all } n \geq 1\right\},$$

   and show that $b(x)$ converges for all $x \in E_\beta$.

   (c) Show that for appropriate β, E_β is non-empty, and in fact is uncountable. [Hint: Consider $m(E_\beta^c)$.]

2. Construct a bounded, monotone increasing function on $\mathbb{R}$ whose set of discontinuities is precisely $\mathbb{Q}$. Can you make the function strictly increasing?

3. (a) Give definitions of each the following types of convergence:

- convergence in measure.
- uniform convergence;
- L^1 convergence;
- almost everywhere convergence;

(b) Indicate without proof which types of convergence imply the others in the form $\mathcal{A} \implies \mathcal{B} \implies \mathcal{C} \implies \mathcal{D}$.

(c) Provide examples of sequences which show that none of these notions of convergence are equivalent. That is, find a sequence that converges $\mathcal{C}$ but not $\mathcal{B}$, etc.

4. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for $X := L^2([0,1])$, and for any $f \in X$ define $\widehat{f}(n) := \langle f, e_n \rangle$. Parts (a) and (b) of this problem are unrelated.

(a) For any $f \in X$ and $p \in [1, \infty)$, define

$$\left\|\widehat{f}\right\|_p = \left(\sum_{n=1}^{\infty} |\widehat{f}(n)|^p \right)^{1/p},$$

whenever the sum is finite, and $\left\|\widehat{f}\right\|_\infty = \sup_{n \geq 1} |\widehat{f}(n)|$. Show that if $\left\|\widehat{f}\right\|_p$ is finite, then so is $\left\|\widehat{f}\right\|_q$ for $q \in [p, \infty]$, and $\left\|\widehat{f}\right\|_q \leq \left\|\widehat{f}\right\|_p$.

(b) Assume $|e_n(x)| \leq M$ for all n and $x \in [0,1]$, and for $f \in X$ and $\alpha > 0$ define

$$|||f|||_{(\alpha)} := \left(\sum_{n=1}^{\infty} n^{2\alpha} |\widehat{f}(n)|^2 \right)^{1/2}$$

whenever this sum is convergent. Show that if $|||f|||_{(\alpha)}$ is finite for some $\alpha > \frac{1}{2}$ then $f \in L^\infty([0,1])$ with $\|f\|_{L^\infty([0,1])} \leq C|||f|||_{(\alpha)}$ for some constant C which is *independent* of f.

[Hints: For (a), first prove it for $q = \infty$; for (b), write f in terms of $\{e_n\}$ and use Cauchy-Schwarz.]

5. Let H be a Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$, and let $A = \{x_n\}_{n=1}^{\infty}$ be an orthonormal set in X.

(a) Give an example of a Hilbert space H and an *infinite* orthonormal set A such that A is not a basis for H.

(b) Prove that if $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2$ for all $x \in H$, then A is a basis for H.

(c) Strengthen the previous part by showing that A is a basis for H provided $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2$ holds for all x in a dense subset of H.

6. The two parts of this problem are unrelated.

(a) Let $X = (0, \infty)$, let $p \geq 1$ be given, and define

$$T(f)(x) := \frac{1}{x^{1/p}} \int_0^x f(y)\, dy.$$

Prove that if $\frac{1}{p} + \frac{1}{q} = 1$, then T is a bounded linear map from $L^q(X)$ to $BC(X)$. Here $BC(X)$ denotes the space of bounded continuous functions on X with the uniform norm $\|g\|_u = \sup_{x \in X}\{|g(x)|\}$.

(b) Compute the limit

$$\lim_{n \to \infty} \int_0^{\infty} \frac{n^2(1 - e^{-x^2/n^2})}{x^2(1 + x^2)}\, dx,$$

with justification where necessary.

7. For any $f \in L^1(\mathbb{R}^d)$ define the maximal function f^* by

$$f^*(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

where the supremum is taken over balls containing x.

(a) Show that if f is not identically zero, then f^* is not integrable. [Hint: Show $|f^*(x)| \geq C|x|^{-d}$ for large $|x|$, by finding a ball with $\int_B |f| > 0$.]

(b) Now suppose f is supported in the unit ball with $\|f\|_{L^1} = 1$. Show that there is a constant $c > 0$ such that

$$m\big(\{x \,:\, f^*(x) > \alpha\}\big) \geq c\,\alpha^{-1}$$

for all sufficiently small α, where m denotes Lebesgue measure.

8. (a) Show that a linear functional on a normed vector space X is bounded if and only if its kernel is closed.

   (b) Prove that given a linear subspace Z of a normed vector space X and $y \in X$ with $d(y, Z) = \delta$ (d the distance function), there exists $\phi \in X^*$ satisfying

$$\|\phi\| \leq 1, \quad \phi(y) = \delta, \quad \text{and} \quad \phi(z) = 0 \quad \text{for all} \quad z \in Z.$$

   If you use a well-known theorem, be sure to clearly identify it.