UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Basic Exam - Statistics
Wednesday, August 28, 2019

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level. Each answer is worth approximately the same number of points.

1. Let $X_1, \ldots, X_n$ be a random sample drawn from a Poisson distribution with mean λ, denoted as $X_i \sim Poisson(\lambda)$,

$$f(x \mid \lambda) = \exp(-\lambda)\frac{\lambda^x}{x!},$$

where $\lambda > 0$ is an unknown parameter and $i = 1, \ldots, n$.

   (a) Write the likelihood for λ and find the maximum likelihood (ML) estimator for λ.

   (b) Is the ML estimator in (a) complete and sufficient? why or why not?

   (c) Is the ML estimator in (a) the uniform minimum variance unbiased estimator (UMVUE) for λ? why or why not?

   (d) Construct a $100(1-\alpha)\%$ confidence interval for λ.

2. Let $X_1, \ldots, X_n$ be a random sample drawn from a Poisson distribution with mean λ, denoted as $X_i \sim Poisson(\lambda)$ where $\lambda > 0$ is an unknown parameter and $i = 1, \ldots, n$. Suppose λ is distributed as Gamma distribution, denoted as $\lambda \sim G(\alpha, \beta)$,

$$f(\lambda \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\lambda^{\alpha-1}\exp\left(-\frac{\lambda}{\beta}\right)$$

where $\alpha > 0, \beta > 0$, and $\Gamma(\cdot)$ denotes the gamma function. Note that the mean and mode of the Gamma distribution are $\alpha\beta$ and $(\alpha-1)\beta$ for $\alpha \geq 1$, respectively.

   (a) Find the posterior distribution of λ.

   (b) Compute the posterior mean of λ and show that the posterior mean is the weighted average of the prior mean and the MLE of λ. When $n \to \infty$, explain the behavior of the posterior mean.

   (c) Find the posterior mode of λ and show that the posterior mode is the weighted average of the prior mode and the MLE of λ. When $n \to \infty$, explain the behavior of the posterior mode.

   (d) Find a $100(1-\alpha)\%$ Bayesian posterior (credible) interval for λ.

3. Let $X_1, \ldots, X_n$ be a random sample drawn from a Normal distribution with unknown mean μ_i and unknown variance θ, and $Y_1, \ldots, Y_n$ be a random sample drawn from a Normal distribution with unknown mean μ_i and unknown variance θ where $i = 1, \ldots, n$. Note that $E(X_i) = E(Y_i) = \mu_i$ and $Var(X_i) = Var(Y_i) = \theta > 0$. The unknown parameters are $\theta, \mu_1, \ldots, \mu_n$ and the parameter of interest for inference is θ.

   (a) Obtain the maximum likelihood (ML) estimators of θ and μ_i, denoted as $\hat{\theta}$ and $\hat{\mu}_i$.

   (b) Show that $\hat{\theta}$ is not consistent for θ. [Hint] : $X_i - Y_i$ is normally distributed with mean 0 and variance 2θ and so $\frac{(X_i - Y_i)^2}{2\theta}$ is chi-squared distributed with degree of freedom 1.

   (c) Construct a consistent estimator for θ using the MLE, $\hat{\theta}$.

   (d) Construct a $100(1-\alpha)\%$ confidence interval for θ using the MLE, $\hat{\theta}$. [Hint] : $\sum_{i=1}^{n} \frac{(X_i - Y_i)^2}{2\theta}$ is chi-squared distributed with degree of freedom n.

4. In a large city, 50% of the population is black. Prospective jurors for court trials are selected from this population. For each selection of a juror, π denotes the probability that a black person is selected.

   (a) Let Y_i be the random variable indicting whether the selected juror i is black, $i = 1, \cdots, 12$. Show that $Y = \sum_{i=1}^{12} Y_i$ is proportional to the likelihood ratio test statistic for testing $H_0 : \pi = 0.5$ vs $H_a : \pi \neq 0.5$.

   (b) Define the P-value based on Y. A supposedly random sampling of 12 prospective jurors contains 1 black person. Find the exact value (as opposed an approximation) of the P-value.

   (c) The use of Bayes factors is a Bayesian alternative to classical hypothesis testing. The Bayes factor is defined as

   $$K = \frac{P(Y|H_0)}{P(Y|H_a)}.$$

   Assuming a uniform prior for π on [0,1], calcuate the Bayes factor for the same hypothesis as above. [Hint 1] : $\int_0^1 x^n(1-x)^m dx = n!m!/(n+m+1)!$, [Hint 2] : $P(Y|H) = \int P(Y|\pi \text{ in } H) f(\pi) d\pi$.