UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Basic Exam - Probability
Friday, August 30, 2019

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level. Each answer is worth approximately the same number of points.

1. Let $X_1, \ldots, X_n$ be iid Uniform(0, θ) with $\theta > 0$. Define $X_{(n)} = max_{i=1,\cdots,n} X_i$.
   (a) Find the probability density function of $X_{(n)}$ and the mean and variance of $X_{(n)}$.
   (b) Find the conditional probability density function of X_i given $X_{(n)} = a$ for some $0 < a < \theta$.
   (c) Define convergence in probability and show that $X_{(n)}$ converges to θ in probability.

2. Let $Y_i, i = 1, \cdots, n$ be Bernoulli random variables. Suppose that $P(Y_i = 1|\pi_i) = \pi_i$ where π's are independent from some probability density function $g(\cdot)$ on $[0, 1]$.
   (a) Explain why $Y = \sum_{i=1}^n Y_i$ has a Binomial distribution unconditionally but not conditionally on $\pi_i, i = 1, \cdots, n$.
   (b) Find the mean and variance of the Binomial distribution.

3. Suppose someone chooses an integer X between 1 and 100 and each integer is equally likely. After that, she chooses an integer Y between 1 and x where each integer is again equally likely. (Note: in both cases, the endpoints are possible choices.)
   (a) Find the mean of Y.
   (b) Find the variance of Y.
   (c) Prove that $Pr(Y \geq t) \leq E(Y)/t$ where $t > 0$. (Hint: $E(Y) = E(Y|Y \geq t)Pr(Y \geq t) + E(Y|Y < t)Pr(Y < t)$.)

4. A Poisson random variable Y with mean $\beta > 0$ has pmf

$$Pr(Y = y) = \frac{e^{-\beta}\beta^y}{y!}, y = 0, 1, \ldots$$

   (a) Prove that $\sum_{x=0}^{\infty} \alpha^x/x! = e^\alpha$.
   (b) Find the MGF of Y.
   (c) Suppose that X is Poisson with mean γ, and X and Y are independent. Find the distribution of $X + Y$.
   (d) Suppose Z has an exponential distribution with pdf $f(z) = \beta e^{-\beta z}, z > 0$. Show that $Pr(Z \leq z) = Pr(Y' > 1)$ where Y' is Poisson with mean $z\beta$.
   (e) Suppose Y_1 and Y_2 are independent Poisson random variables with the same means. Show that $Y_1|Y_1 + Y_2 = 7$ is Binomial(7,0.5).