Department of Mathematics and Statistics
University of Massachusetts

Topology qualifying exam

Wednesday, August 28, 2019

Answer all seven questions. Justify your answers.
Passing standard: 70% with four questions essentially complete.

1. Let X, Y be topological spaces. Prove the following:
   (a) For any subset $A \subset X$, we have $\mathrm{Int}(X \setminus A) = X \setminus \overline{A}$.
   (b) A map $f\colon X \to Y$ is continuous $\iff$ $f^{-1}(\mathrm{Int}A) \subset \mathrm{Int}(f^{-1}(A))$ for all $A \subset Y$.

2. For A, B any subsets of $\mathbb{R}^n$, let $d(A,B) = \inf\{|x-y| \mid x \in A,\ y \in B\}$ be the *distance* between A and B. Show that if A is compact, B is closed, and $A \cap B = \emptyset$, then the distance between A and B is positive.

3. Let $q\colon M \to N$ be a finite (connected) covering.
   (a) Prove that if N is a compact n–dimensional manifold, then so is N.
   (b) Tell which spaces (up to homeomorphism) finitely cover the Klein bottle. *(You can use the classification of compact surfaces.)*

4. (a) Let a be a circle in $\mathbb{RP}^2$, which lifts to the equator in its covering space S^2. Let b be the boundary circle of the Möbius band B. Let $X = \mathbb{RP}^2 \underset{a\sim b}{\cup} B$ be the space obtained by gluing these two spaces along the circles a and b. Compute $\pi_1(X)$.
   (b) Show that *any* cover of the product space $Y = \mathbb{RP}^2 \times B$ is a normal cover.

5. Let M be an n-dimensional manifold, and let x be any point of M. Consider the inclusion induced homomorphism $\phi : H_i(M \setminus \{x\}) \to H_i(M)$.
   (a) Prove that ϕ is an isomorphism for $i < n-1$ and is surjective for $i = n-1$.
   (b) If M is also compact and orientable, show that ϕ is an isomorphism for $i = n-1$ as well. Illustrate that for a compact non-orientable M, this does not need to hold.

6. Let X be the quotient $(S^1 \times I)/\sim$ by the following equivalence relation $\sim$:
   - $(z,0) \sim (w,0)$ if and only if $z^4 = w^4$
   - $(z,1) \sim (w,1)$ if and only if $z^6 = w^6$
   - in all other cases, $(z,s) \sim (w,t)$ only when $z = w$ and $s = t$.

   Compute the homology and cohomology groups $H_i(X;G)$ and $H^i(X;G)$, with coefficients $G = \mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$, respectively.

7. Show that $\mathbb{RP}^n$ does not retract onto $\mathbb{RP}^k$ for any $n > k > 0$.