DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST

ALGEBRA EXAMINATION

AUGUST 2019

Passing Standard: To pass the exam it is sufficient to solve five problems including a least one problem from each of the three parts. Show all your work and justify your answers carefully. All rings contain identity and all ring homorphisms preserve the identity.

1. Group theory

1. Show that there are no simple groups of order 80.

2. Let p be a prime and let P be a group of order p^a, $a \geq 1$.
(1) Show that the center $Z(P)$ of P is nontrivial.
(2) Let H be a nontrivial normal subgroup of P. Show that H has nontrivial intersection with $Z(P)$.

3. Let G be a free abelian group of rank r: $G \cong \mathbf{Z}^r$. Fix $n \geq 1$. Show that G has finitely many subgroups of index n.

2. Ring theory

4. Prove that every prime ideal in $\mathbf{Z}[x]$ can be generated by at most two elements.

5. Let F be a field of characteristic different from 2. Let V be a vector space over F. Show that $V \otimes_F V \cong \mathrm{Sym}^2(V) \oplus \wedge^2(V)$.

6. Let R denote the ring $\mathbf{Z}[\sqrt{-5}]$ and $\mathfrak{p}$ the ideal $(3, 1+\sqrt{-5})$ in R.
(1) Show that $\mathfrak{p}$ is a prime ideal.
(2) Show that $\mathfrak{p}$ is not a principal ideal. (You may find the multiplicative norm map $N : R \to \mathbf{Z}$ given by $N(a + b\sqrt{-5}) = a^2 + 5b^2$ helpful.)
(3) Let S be the complement of $\mathfrak{p}$ in R. Show that the ideal $S^{-1}\mathfrak{p}$ is principal in the localization $S^{-1}R$.

3. Field theory

7. Let $f(x) = x^4 + x^2 - 2x + 1$.
(1) Show that f is irreducible over $\mathbf{Q}$.
(2) Determine the Galois group of the splitting field $F/\mathbf{Q}$ of f.
(Recall that if $g = x^4 + px^2 + qx + r$, then the discriminant of g is given by

$$16p^4r - 4p^3q^2 - 128p^2r^2 + 144pq^2r - 27q^4 + 256r^3$$

and the resolvent cubic of g is $x^3 - 2px^2 + (p^2 - 4r)x + q^2$.)

8. Let $\alpha = \sqrt{3 + \sqrt{2}} \in \mathbf{C}$.
(1) Find the minimal polynomial f of α over $\mathbf{Q}$, with proof.

(2) Let K denote the splitting field of f over $\mathbf{Q}$. Determine the Galois group $\mathrm{Gal}(K/\mathbf{Q})$.

9. Let $\alpha \in \mathbf{C}$ be a root of the polynomial $f(x) = x^3 + 4x + 2$. Show that α is not contained in $\mathbf{Q}(\zeta_n)$ for any $n \geq 1$, with ζ_n a primitive nth root of unity.