Advanced Calculus/Linear algebra basic exam

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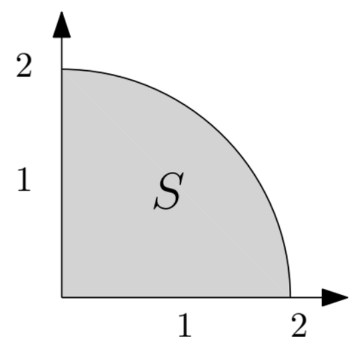
Instructions: Do 7 of the 8 problems. Show your work. The passing standards are:

- Master's level: 60% with three questions essentially, complete (including one question from each part);
- Ph.D. level: 75% with two questions from each part essentially complete.

Advanced Calculus

1. Compute the following integrals and limits.
   (a) $\int_1^2 \frac{\ln x}{x^2} dx$
   (b) $\int \frac{x-3}{\sqrt{x^2-6x}} dx$
   (c) $\iint_S \sqrt{4-x^2-y^2}\,dxdy$ where S is the region:



2. (a) Find the limit $\lim_{x\to 0} \frac{e^{x^2}-1}{2x^2}$ in two ways.
   (b) Consider the power series $\sum_{n=0}^{\infty} a_n z^n$ where $a_0 = 0, a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. Find the radius of convergence of the power series.

3. Find the minimum and maximum values of the function $f(x, y) = xy$ on the region $S = \{(x, y) \mid x^2 + y^2/4 \leq 1\}$. On the boundary of the region, use the method of Lagrange multipliers.

4. Let $f(x, y) = \frac{-y}{x^2 + y^2}$ for $(x, y) \neq 0$.
   (a) Determine if $\lim_{(x,y)\to(0,0)} f(x, y)$ exists, explain your answer. If the limit exists, compute it.
   (b) Let $g(x, y) = \frac{x}{x^2 + y^2}$. Show that $\oint_{\partial S} (f(x, y)dx + g(x, y)dy) = 2\pi$ if S is any open set in the plane containing $(0, 0)$ and ∂S is its boundary.

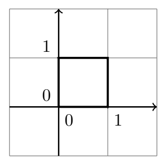
Turn the page

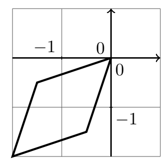
Linear Algebra

1. (a) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & z \\ 1 & -4 & 7 \end{bmatrix}$. Find all real values of z for which $A\mathbf{x} = \mathbf{b}$ has one unique solution for every $\mathbf{b} \in \mathbb{R}^3$.

   (b) Let z be any value you found in (a) and let $A' = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & z & 3 \\ 1 & -4 & 7 & 5 \end{bmatrix}$. Using **part (a)**, determine for which $\mathbf{b} \in \mathbb{R}^3$ the system $A'\mathbf{x} = \mathbf{b}$ has no solution, a unique solution and infinitely many solutions.

2. (a) Consider how a linear transformation T acts on the following picture. Find one of the eigenvectors and the corresponding eigenvalue.

   Original: 

   After T: 

   (b) Let $A = \begin{bmatrix} 3 & 6 \\ 2 & y \end{bmatrix}$. Find all values of y such that one of the eigenvalues of A is 0.

   (c) Let $B = \begin{bmatrix} 1 & z \\ 4 & 3 \end{bmatrix}$. Find all values of z such that the linear transformation T determined by B fixes no line in $\mathbb{R}^2$, i.e. T sends no line in $\mathbb{R}^2$ to itself.

3. (a) Let A and B be $n \times n$ matrices. If $AB = 0$ (where 0 is the zero-matrix), show that $\text{rank}(A) + \text{rank}(B) \leq n$.

   (b) For any $n \times n$ matrix A, show that there exists a $n \times n$ real matrix B with $AB = 0$ and $\text{rank}(A) + \text{rank}(B) = n$.

4. (a) Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T_1(\mathbf{v}) = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \mathbf{v}$ for every $\mathbf{v} \in \mathbb{R}^2$. Prove that for any linear transformation $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$, the linear transformation $T_2 \circ T_1$ is never surjective (onto).

   (b) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Show that $\mathbb{R}^n \supseteq \text{Im}(T) \supseteq \text{Im}(T^2) \supseteq \text{Im}(T^3) \supseteq \ldots$, and show that there exists some $m \in \mathbb{N}^+$ such that $\text{Im}(T^k) = \text{Im}(T^{k+1})$ for all $k \geq m$.