DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
ADVANCED DIFFERENTIAL EQUATIONS EXAM
AUGUST 2018

This exam consists of eight questions, equally weighted. You are encourage to try to solve all questions, and to pass you must score at least 65% with at least five questions essentially correct.

1. Solve the initial value problem

$$\partial_{tt}u - \partial_{xx}u = \cos(x)$$
$$u(x,0) = \sin(x), \quad \partial_t u(x,0) = \cos(x)\,.$$

2. Use method of characteristics to solve the equation

$$u_x + u_y + u = e^{x+2y}$$
$$u(x,0) = 0\,.$$

3. A function $u \in C^2(\bar{\Omega})$ is *subharmonic* if $-\Delta u \leq 0$ in Ω.

   a If $u \in C^2(\bar{\Omega})$ is subharmonic, show that

$$u(x) \leq \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)\mathrm{d}y$$

   for all $B(x,r) \subset \Omega$.

   b State and prove the maximum principle for subharmonic functions.

4. Consider the wave equation with a higher-order damping

$$\partial_{tt}u - c^2\partial_{xx}u - \gamma\partial_{xxt}u = 0 \qquad \text{in } 0 < x < 1, \quad t > 0$$

   along with the boundary condition $\partial_x u(0,t) = \partial_x(1,t) = 0$ for all $t > 0$, where $\gamma > 0$ is the damping coefficient.

   a Define a quadratic functional $E(u)$ that represents the energy associated with such waves, and show that

$$\frac{\mathrm{d}E(u(x,t))}{\mathrm{d}t} \leq 0$$

   for any solution $u(x,t)$.

b Formulate the initial-value problem for this PDE together with its boundary conditions, and use (a) to prove the uniqueness of solutions.

5. Let

$$A = \begin{bmatrix} -1 & 10 \\ 0 & -1 \end{bmatrix}.$$

(a) Determine the stability of the origin for the linear system

$$x'(t) = Ax(t).$$

(b) Determine the stability of the origin for the 2-periodic, piecewise linear system:

$$x'(t) = \begin{cases} Ax(t), & 2k < t < 2k+1, k \in \mathbb{Z} \\ A^T x(t), & 2k+1 < t < 2k+2, k \in \mathbb{Z} \end{cases}$$

6. Let

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 + y_2 - y_2(y_1^2 + 2y_2^2) \end{aligned}$$

Prove that there exists a nontrivial periodic solution. (Hint: a circle of sufficiently large radius is forward-invariant.)

7. Show that the origin is stable for the system

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -y_1^3 - y_2 H(y_2), \end{aligned}$$

where

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases}$$

(Hint: find a Lyapunov function.)

8. Consider the ODE system

$$x' = Ax + g(x), \qquad \text{for } x \in \mathbb{R}^n$$

where $g(x)$ is a smooth function and $|g(x)| = O(|x|^2)$ as $x \to 0$. Suppose the matrix A has one positive eigenvalue. Prove that the origin is unstable.