DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERIC ANALYSIS EXAM
AUGUST 2017

Do five of the following problems. All problems carry equal weight.
Passing level:
Masters: 60% with at least two substantially correct.
PhD: 75% with at least three substantially correct.

1. Consider $f(x) = 2 - x + x^2 - x^3$. Let $p_2(x)$ denote the second order polynomial interpolation of $f(x)$ at $\{-1, 0, 1\}$.

   (a) Find $p_2(x)$.

   (b) Compute the L^∞ error of $p_2(x)$ on the domain $[-1, 1]$.

2. Which of the following iterations will converge to the indicated fixed point α (provided x_0 is sufficiently close to α)? If it does converge, give the order of the convergence; for linear convergence, give the rate of linear convergence.

   (a) $x_{n+1} = -1 + x_n + \frac{2}{x_n}, \quad \alpha = 2$

   (b) $x_{n+1} = \frac{1}{2}x_n + \frac{3}{2x_n}, \qquad \alpha = \sqrt{3}$

   (c) $x_{n+1} = \frac{16}{1+x_n} - 1, \quad \alpha = 3$

3. Consider the numerical solution of $y' = f(y)$ with a scheme

$$y_{n+1} = y_n + \frac{h}{2}[f(y_n) + f(y_n + hf(y_n))] \,.$$

   (a) Find the local truncation error of this numerical scheme.

   (b) Assume $f(y) = -My$ for some large $M > 0$. Find the range of h such that this numerical scheme is stable.

   (c) Find an unconditionally stable numerical scheme for the above $f(y)$.

4. Let $\mathbf{A}$ be a real $n \times n$ non-singular matrix. Let $b_N, \cdots, \mathbf{b}_N$ be N vectors in $\mathbb{R}^n$.

   (a) Sketch an efficient algorithm that solves N linear systems

$$\mathbf{A}\mathbf{x}_1 = \mathbf{b}_1, \cdots, \mathbf{A}\mathbf{x}_N = \mathbf{b}_N \,.$$

   (b) Count the number of Additions/Subtractions and Multiplications/Divisions.

5. Consider the numerical integration rule

$$I(f) = \int_{-h}^{h} f(x)\,dx \approx A_0 f(-h/2) + A_1 f(0) + A_2 f(h/2).$$

   (a) Find A_0, A_1, and A_2 such that the integration rule is exact for polynomials of degree ≤ 2.

   (b) Show that the rule constructed in (a) is in fact exact for polynomials of degree ≤ 3.

(c) For the constructed rule, it can be proved that

$$I(f) - [A_0 f(-h/2) + A_1 f(0 + A_2 f(h/2)] = c_0 f^{(4)}(\eta) h^5, \ \ \eta \in (-h, h)$$

where c_0 is a constant independent of f. Find the constant c_0.

6. Find the values of a and b which solve the following optimization problem:

$$\min_{a,b} \int_0^\infty (x^2 - ax - b)^2 \, e^{-x} \, dx.$$

Note that the function $f(x) = (ax + b)$ is the weighted L^2 projection of x^2 onto the space spanned by $\{1, x\}$.

7. Let $\mathbf{A}$ be an $n \times n$ matrix. The numerical radius of $\mathbf{A}$ is defined as

$$r(\mathbf{A}) = \max_{\mathbf{x}^T\mathbf{x} \le 1} \left|\mathbf{x}^T \mathbf{A} \mathbf{x}\right|.$$

(a) Show that $r(\mathbf{A}) \le \|\mathbf{A}\|_2$. (Hint: for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $|\mathbf{x}^T \mathbf{y}| \le \|\mathbf{x}\|_2 \|\mathbf{y}\|_2$.)

(b) Show that $\|\mathbf{A}\|_2 \le 2r(\mathbf{A})$. (Hint: Let $\mathbf{A} = (\mathbf{A} + \mathbf{A}^T)/2 + (\mathbf{A} - \mathbf{A}^T)/2 := \mathbf{A}_1 + \mathbf{A}_2$. $\mathbf{A}_1$ is symmetric and $\mathbf{A}_2$ is anti-symmetric.)

(c) The condition number $\kappa(\mathbf{A}) = \|\mathbf{A}\|_2 \cdot \|\mathbf{A}^{-1}\|_2$. Show that $\kappa(\mathbf{A}) \ge 1$.