DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - STATISTICS
Fall 2016

Work all problems. Show all work. Explain your answers. Sixty points are needed to pass at the Master's level and seventy five at the Ph.D. level.

1) Let $X_1, X_2, ..., X_n$ be a random sample from a $\text{Normal}(\mu, 1)$ distribution. You are given that the normal density is: $f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ \frac{-(x-\mu)^2}{2\sigma^2} \right\}$.

a) (5 points) State the likelihood of μ.
b) (5 points) Find the maximum likelihood estimator of μ, $\hat{\mu}^{(MLE)}$.
c) (7 points) Assign a $\text{Normal}(\lambda, \phi^2)$ prior to μ, with the following density:

$$f(\mu) = \frac{1}{\sqrt{2\pi\phi^2}} \exp\left\{ \frac{-(\mu-\lambda)^2}{2\phi^2} \right\}$$

Find the posterior distribution of μ.
d) (6 points) Find the posterior mean of μ.
e) (5 points) Describe how to construct a 95% equal-tail posterior interval for μ.

2) Suppose $X_1, X_2, ..., X_n$ is a random sample from a distribution with the following density:

$$f(x \mid \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}, \quad \text{for } x = 0, 1, ..., n;$$

where $0 \le \theta \le 1$ and n is any positive integer.

a) (4 points) Derive the maximum likelihood estimator of θ, $\hat{\theta}^{(MLE)}$. It is required to justify that your answer is in fact a maximum likelihood estimator.
b) (4 points) Is $\hat{\theta}^{(MLE)}$ an unbiased estimator? Justify your answer.
c) (4 points) Find the variance of the $\hat{\theta}^{(MLE)}$ that you found in part a).
d) (6 points) Assuming that $0 \le \theta \le 1$, find an approximate distribution of the $\hat{\theta}^{(MLE)}$ that you found in part a) as the sample size n gets large.
e) (4 points) Use the result from part d) to find an approximate 95% confidence interval for θ.
f) (5 points) Derive the likelihood ratio test for the testing the following 2 hypotheses:

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta = \theta_1,$$

where $0 < \theta_0 < \theta_1 < 1$. For this, you are required to reduce the test in terms of a statistic with a known distribution, and then describe the size α rejection region using this known distribution.

3) Suppose $X_1, X_2, ..., X_n$ is a random sample from the Poisson(λ) distribution, and let $\theta = \lambda^2$.

a) (7 points) Derive the Cramer-Rao bound on an unbiased estimator of λ.
b) (7 points) Determine the Cramer-Rao bound on an unbiased estimator of $\theta = \lambda^2$.
c) (7 points) Determine the Uniform Minimum Variance Unbiased Estimator of $\theta = \lambda^2$.

4) Suppose $X_1, X_2, ..., X_n \overset{i.i.d.}{\sim}$ Uniform$(0, \theta)$. You wish to test the null hypothesis $\theta = \theta_0 = 1$ against the alternative $\theta < 1$ (so the full parameter space under consideration is $\Theta = (0,1]$).

a) (4 points) Give the form of the likelihood ratio test for this problem.
b) (4 points) Is this a UMP-level α test? Why or why not?
c) (4 points) Suppose $n = 1$, so the test is based on a single observation, X. Find the critical value of a level 0.05 test.
d) (4 points) Sketch the power function of the test in the previous part, over the range $\Theta = (0,1]$. Compute and label the power at $\theta = 0.05$, $\theta = 0.5$, and $\theta = 1$.
e) (4 points) Suppose $X = 0.1$ is observed. What is the p-value?
f) (4 points) Now suppose n > 1. Suppose you found an appropriate $\alpha = 0.05$ level likelihood ratio test. Would the type I and type II errors be higher or lower than in the test with $n = 1$?