UMASS Amherst

Department of Mathematics and Statistics

Probability Basic Exam

Fall 2016

Work all problems. 60 points are needed to pass at the Masters level and 75 to pass at the Ph.D. level. A Ph.D. pass must include at least 8 points on each of the four problems.

(1) (a) Let random variables X, Y, and Z be distributed according to the following table.

x	y	z	probability
-1	-1	0	1/8
-1	1	0	1/8
1	-1	0	1/8
1	1	0	1/8
-2	-2	1	1/8
-2	2	1	1/8
2	-2	1	1/8
2	2	1	1/8

i. [8 pts] **True** or **False**: X and Y are independent. Explain.

ii. [8 pts] **True** or **False**: X and Y are conditionally independent given Z. Explain.

(b) Let random variables X, Y, and Z be distributed according to the following table.

x	y	z	probability
-1	-1	0	1/4
1	1	0	1/4
-1	1	1	1/4
1	-1	1	1/4

i. [8 pts] **True** or **False**: X and Y are independent. Explain.

ii. [8 pts] **True** or **False**: X and Y are conditionally independent given Z. Explain.

(2) Chebychev's Inequality gives an upper bound on how far a random variable can vary from its mean. Specifically, the inequality states that for any random variable X with mean μ and SD σ and for any $\epsilon > 0$, $\Pr[|X - \mu| \geq \epsilon] \leq \sigma^2/\epsilon^2$.

(a) [8 pts] Prove Chebychev's Inequality for a continuous random variable X. Hint: Start by writing the definition of variance as an integral.

The rest of this problem investigates conditions under which the bound is tight, i.e., conditions under which $\Pr[|X - \mu| \geq \epsilon] = \sigma^2/\epsilon^2$. Suppose that μ and σ are given. We'll investigate Chebychev's Inequality first for $\epsilon = \sigma$ and then for $\epsilon \neq \sigma$.

(b) [8 pts] Let $\epsilon = \sigma$. Are there any continuous random variables X with mean μ and SD σ such that $\Pr[|X - \mu| \geq \epsilon] = \sigma^2/\epsilon^2$? If not, explain. If so, show how to construct one.

(c) [8 pts] Let $\epsilon = \sigma$. Are there any discrete random variables X with mean μ and SD σ such that $\Pr[|X - \mu| \geq \epsilon] = \sigma^2/\epsilon^2$? If not, explain. If so, show how to construct one.

(d) [8 pts] Now suppose $\epsilon \neq \sigma$. Are there any random variables X with mean μ and SD σ such that $\Pr[|X - \mu| \geq \epsilon] = \sigma^2/\epsilon^2$? If not, explain. If so, show how to construct one.

(3) [12 pts] Suppose X and Y each have marginal $\mathcal{N}(0, 1)$ distributions and $\text{Cov}(X, Y) = 0$. Are X and Y necessarily independent? If so, provide a proof. If not, provide a counterexample.

(4) Let $X_1, \ldots, X_n$ be i.i.d. uniform on the interval $(0, \theta)$ with $\theta > 0$. Define $T_n = \max\{X_1, \ldots, X_n\}$.

(a) [8 pts] Find $\Pr(|T_n - \theta| \geq \epsilon)$ for any fixed $\epsilon > 0$.

(b) [8 pts] Show that T_n converges to θ in probability.

(c) [8 pts] Find the range of $\gamma(> 0)$ such that $n^\gamma\{T_n - \theta\} \xrightarrow{p} 0$ as $n \to \infty$.