University of Massachusetts  
Department of Mathematics and Statistics  
Advanced Exam in Geometry  
For August, 2016

Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. *Passing standard: 70% with three problems essentially complete.* Justify all your answers.

1. Let $E \rightarrow \mathbb{RP}^2$ be the tautological line bundle (i.e., $\forall x \in \mathbb{RP}^2$, the fiber $E_x$ is the 1-dimensional subspace of $\mathbb{R}^3$ represented by $x$) and let $E' \rightarrow \mathbb{RP}^2$ be the rank 2 bundle whose fiber at $x \in \mathbb{RP}^2$ is the 2-dimensional subspace of $\mathbb{R}^3$ that is orthogonal to the line represented by $x$. Show that $E \oplus E' \rightarrow \mathbb{RP}^2$ is isomorphic to the product bundle $\mathbb{RP}^2 \times \mathbb{R}^3$ as smooth vector bundles.

2. Let $E \rightarrow S^1$ be the nontrivial rank 1 real vector bundle over the circle, e.g., $E = \mathbb{R} \times \mathbb{R}/\{(x,y) \sim (x+1,-y)\}$, and let $M$ be the set defined by

$$M := \sqcup_{x \in S^1} P(E_x \oplus \mathbb{R}),$$

where $P(E_x \oplus \mathbb{R})$ is the space of 1-dimensional subspaces of the 2-dimensional vector space $E_x \oplus \mathbb{R}$.

(a) Show that $M$ is a 2-dimensional smooth manifold.
(b) Determine whether $M$ is orientable and explain why.

3. Let $M$ be the smooth 3-manifold obtained by identifying $\{0\} \times S^2$ and $\{1\} \times S^2$ in $[0,1] \times S^2$ via the map $(0,x) \mapsto (1,-x)$ for any $x \in S^2 \subset \mathbb{R}^3$. Compute the de Rham cohomology groups of $M$.

4. Let $(M,g)$ be a 2-dimensional Riemannian manifold, and let $\nabla$ be the Levi-Civita connection. For any point $x \in M$, define

$$K(x) \equiv \frac{\langle R(X,Y)Y,X \rangle}{\sqrt{\vert X \vert^2 \vert Y \vert^2 - \langle X,Y \rangle^2}},$$

where $X,Y \in T_xM$ is a pair of linearly independent vectors. (Here $R(X,Y)Z \equiv \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$ is the curvature endomorphism.) Show that

(1) $K(x)$ depends only on $x$ (i.e., independent of the choice of $X,Y$).
(2) If $K \equiv 0$ on $M$, then $g$ is locally isometric to the Euclidean metric.
5. Let $\pi : T^*M \to M$ be the cotangent bundle of the smooth manifold $M$. We define a 1-form $\tau$ on $T^*M$ as follows: for any $p \in M$, $v \in T^*_pM$, the value $\tau(p,v) \in T^*_p(T^*M)$ at $(p,v)$ is given by $\pi^*(v)$, where $\pi^* : T^*_pM \to T^*_{(p,v)}(T^*M)$ is the dual of $\pi^* : T_{(p,v)}(T^*M) \to T_pM$. Show that $\tau$ is a smooth 1-form and $\omega = -d\tau$ is a symplectic structure on $T^*M$. (A symplectic structure by definition is a closed, non-degenerate 2-form.)

6. Let $G \subset GL(2, \mathbb{R})$ be the set of all $2 \times 2$ matrices $A$ such that $A^tQA = Q$, where $Q$ is the diagonal matrix with entries 1 and $-1$.

   (a) Show that $G$ is a Lie group, and determine its Lie algebra and calculate its dimension.

   (b) How many components does $G$ have?

   (c) Give an explicit parametrization of the identity component of $G$ via the exponential map.

7. Let $(S,g)$ be a parameterized Riemannian surface with local coordinates $(u,v)$. We say $(S,g)$ is diagonal if the metric is a diagonal matrix in these coordinates, that is $g_{12} = g_{21} = 0$ for all $u,v$.

   (a) Compute the Christoffel symbols $\Gamma^k_{ij}$ in these coordinates.

   (b) Write down the geodesic equations in these coordinates.

   (c) Show that the isometrically embedded surface

   $$\{(u \cos v, u \sin v, u)/\sqrt{2} \mid u > 0, 0 \leq v < 2\pi\} \subset \mathbb{R}^3$$

   is diagonal, and that

   $$u = A \sec(v/\sqrt{2} + B)$$

   is a geodesic, where $A, B$ are constants.

8. Let $X,Y \in \mathcal{X}(\mathbb{R}^3)$ be defined by

   $$X = y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \quad Y = z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}.$$ 

   (a) Find the maximal subset $U$ of $\mathbb{R}^3$ on which $X,Y$ determine a 2-dimensional distribution $\Delta$.

   (b) Show that $\Delta$ is integrable on $U$.

   (c) Describe the 2-dimensional integral manifold of $\Delta$ through the point $(1,1,1)$. 