University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
For August, 2016

Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

1. Let $E \to RP^2$ be the tautological line bundle (i.e., $\forall x \in RP^2$, the fiber E_x is the 1-dimensional subspace of $\mathbb{R}^3$ represented by x) and let $E' \to RP^2$ be the rank 2 bundle whose fiber at $x \in RP^2$ is the 2-dimensional subspace of $\mathbb{R}^3$ that is orthogonal to the line represented by x. Show that $E \oplus E' \to RP^2$ is isomorphic to the product bundle $RP^2 \times \mathbb{R}^3$ as smooth vector bundles.

2. Let $E \to S^1$ be the nontrivial rank 1 real vector bundle over the circle, e.g., $E = \mathbb{R} \times \mathbb{R}/\{(x,y) \sim (x+1,-y)\}$, and let M be the set defined by

$$M := \sqcup_{x \in S^1} P(E_x \oplus \mathbb{R}),$$

where $P(E_x \oplus \mathbb{R})$ is the space of 1-dimensional subspaces of the 2-dimensional vector space $E_x \oplus \mathbb{R}$.

   (a) Show that M is a 2-dimensional smooth manifold.

   (b) Determine whether M is orientable and explain why.

3. Let M be the smooth 3-manifold obtained by identifying $\{0\} \times S^2$ and $\{1\} \times S^2$ in $[0,1] \times S^2$ via the map $(0,x) \mapsto (1,-x)$ for any $x \in S^2 \subset \mathbb{R}^3$. Compute the de Rham cohomology groups of M.

4. Let (M,g) be a 2-dimensional Riemannian manifold, and let ∇ be the Levi-Civita connection. For any point $x \in M$, define

$$K(x) \equiv \frac{\langle R(X,Y)Y, X\rangle}{\sqrt{|X|^2|Y|^2 - \langle X, Y\rangle^2}},$$

where $X, Y \in T_xM$ is a pair of linearly independent vectors. (Here $R(X,Y)Z \equiv \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ is the curvature endomorphism.) Show that

(1) $K(x)$ depends only on x (i.e., independent of the choice of X, Y).

(2) If $K \equiv 0$ on M, then g is locally isometric to the Euclidean metric.

5. Let $\pi : T^*M \to M$ be the cotangent bundle of the smooth manifold M. We define a 1-form τ on T^*M as follows: for any $p \in M$, $v \in T_p^*M$, the value $\tau(p,v) \in T_{(p,v)}^*(T^*M)$ at (p,v) is given by $\pi^*(v)$, where $\pi^* : T_p^*M \to T_{(p,v)}^*(T^*M)$ is the dual of $\pi_* : T_{(p,v)}(T^*M) \to T_pM$. Show that τ is a smooth 1-form and $\omega = -d\tau$ is a symplectic structure on T^*M. (A symplectic structure by definition is a closed, non-degenerate 2-form.)

6. Let $G \subset GL(2,\mathbb{R})$ be the set of all 2×2 matrices A such that $A^tQA = Q$, where Q is the diagonal matrix with entries 1 and -1.

   (a) Show that G is a Lie group, and determine its Lie algebra and calculate its dimension.

   (b) How many components does G have?

   (c) Give an explicit parametrization of the identity component of G via the exponential map.

7. Let (S,g) be a parameterized Riemannian surface with local coordinates (u,v). We say (S,g) is *diagonal* if the metric is a diagonal matrix in these coordinates, that is $g_{12} = g_{21} = 0$ for all u, v.

   (a) Compute the Christoffel symbols Γ_{ij}^k in these coordinates.

   (b) Write down the geodesic equations in these coordinates.

   (c) Show that the isometrically embedded surface

   $$\{(u\cos v, u\sin v, u)/\sqrt{2} \mid u > 0, 0 \le v < 2\pi\} \subset \mathbb{R}^3$$

   is diagonal, and that

   $$u = A\sec(v/\sqrt{2} + B)$$

   is a geodesic, where A, B are constants.

8. Let $X, Y \in \mathcal{X}(\mathbb{R}^3)$ be defined by

   $$X = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad Y = z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}.$$

   (a) Find the maximal subset U of $\mathbb{R}^3$ on which X, Y determine a 2-dimensional distribution Δ.

   (b) Show that Δ is integrable on U.

   (c) Describe the 2-dimensional integral manifold of Δ through the point $(1,1,1)$.