UMass Amherst Algebra Advanced Exam
Tuesday August 30, 2016, 10AM – 1PM, LGRT 1634

Instructions: To pass the exam it is sufficient to solve five problems including a least one problem from each of the three parts. Show all your work and justify your answers carefully.

1. Group theory and representation theory

Q1. Let G be a finite group of order $|G| = 765 = 3^2 \cdot 5 \cdot 17$. Suppose that there exists a normal subgroup H of G of order $|H| = 85 = 5 \cdot 17$.

(a) Show that H is cyclic.
(b) Compute the order of the automorphism group of H.
(c) Using part (b) or otherwise, show that G is abelian.

Q2. Let q be a prime power and $\mathbb{F}_q$ the finite field of order q. By considering the characteristic polynomial or otherwise, determine the number of conjugacy classes in $\mathrm{GL}_2(\mathbb{F}_q)$.

Q3. For the symmetric group S_4 determine the number of irreducible representations and their dimensions.

2. Commutative Algebra

Q4. Let $\omega = e^{2\pi i/3} = (-1+\sqrt{3}i)/2$. Consider the ring

$$R = \mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}.$$

Prove that R is a unique factorization domain.

Q5. Consider the ring

$$R = \mathbb{Z}[\sqrt{-13}] = \{a + b\sqrt{-13} \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}.$$

Let $P = (7, 1+\sqrt{-13}) \subset R$ be the ideal of R generated by 7 and $1+\sqrt{-13}$.

(a) Show that P is a maximal ideal.
(b) Show that P is *not* a principal ideal.
(c) Write $S = R \setminus P$. Let $R_P = S^{-1}R$ be the localization of R at the prime ideal $P \subset R$, and $P_P = S^{-1}P \subset R_P$ the ideal of R_P obtained from the ideal $P \subset R$ by extension of scalars. Show that P_P is a principal ideal.

Q6. Let $R = \mathbb{C}[x, y]$ be the ring of polynomials in the variables x and y with coefficients in $\mathbb{C}$. Let $M = (x, y) \subset R$ be the maximal ideal of R generated by x and y. In this question we study the R-module $M \otimes_R M$.

(a) Show that there is a well-defined homomorphism of R-modules

$$\theta \colon M \otimes_R M \to R/M = \mathbb{C}$$

given by

$$\theta\left(\sum_{i=1}^{n} f_i \otimes g_i\right) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x}(0,0) \cdot \frac{\partial g_i}{\partial y}(0,0).$$

Here we identify the R-module R/M with $\mathbb{C}$ by the map

$$f(x, y) + M \mapsto f(0, 0).$$

(b) Using part (a) or otherwise, prove that $x \otimes y - y \otimes x$ is a nonzero element of $M \otimes_R M$.

(c) Show that there exists a nonzero element r of R such that

$$r \cdot (x \otimes y - y \otimes x) = 0.$$

3. Field theory and Galois theory

Q7.

(a) Show that the polynomial $x^5 + x^2 + 1$ is irreducible in $(\mathbb{Z}/2\mathbb{Z})[x]$.

(b) Using part (a) or otherwise, show that the polynomial

$$f = x^5 + 4x^3 + 3x^2 + 2x + 7$$

is irreducible in $\mathbb{Q}[x]$.

(c) Let $\alpha \in \mathbb{R}$ be a real root of the polynomial f of part (b). Show that there do *not* exist a positive integer $r \in \mathbb{N}$ and rational numbers $a_1, \ldots, a_r \in \mathbb{Q}$ such that

$$\alpha = \sqrt[3]{a_1} + \cdots + \sqrt[3]{a_r}.$$

Q8. Let K be the splitting field of the polynomial $f = x^4 - 8x^2 + 11$ over $\mathbb{Q}$.

(a) Show that the Galois group G of the field extension $\mathbb{Q} \subset K$ is isomorphic to a subgroup of the dihedral group D_4 of symmetries of the square.

(b) Determine the Galois group G of $\mathbb{Q} \subset K$.

Q9. Let $\mathbb{Q} \subset K$ be a Galois extension with Galois group the alternating group A_5. Let $\alpha \in K$ be an element such that $\alpha \notin \mathbb{Q}$. Prove that the minimal polynomial of α over $\mathbb{Q}$ has degree at least 5.