Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Topology

September 2, 2015

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.

Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

Throughout this exam, $\mathbb{R}$ denotes the real line with the standard topology.

(1) Let X be a path-connected and Hausdorff topological space with a *fixed point free involution* $\tau\colon X \to X$ (i.e. a homeomorphism with $\tau \circ \tau = \mathrm{id}_X$ and $\tau(x) \neq x$ for each point $x \in X$). Let $f\colon X \to Y$ be the projection from X onto the quotient space $Y := X/{\sim}$, where $x' \sim x$ if and only if $x' = x$ or $x' = \tau(x)$. Let $y_0 = f(x_0)$, for $x_0 \in X$.

  (a) Prove that $f\colon X \to Y$ is a covering map.

  (b) Prove that the following are equivalent:

    (i) There exists an element $\alpha \in \pi_1(Y, y_0) \setminus f_*(\pi_1(X, x_0))$ such that α^2 is equal to the identity element of $\pi_1(Y, y_0)$.

    (ii) There exists a path γ in X from x_0 to $\tau(x_0)$ such that $\tau \circ \gamma$ is homotopic relative to its endpoints to the "opposite" path $\bar{\gamma}$ of γ.

(2) Let X be the union of the unit sphere in the 3-space with the straight line segment from the north pole to the south pole. What is $\pi_1(X)$?

(3) Let $X = \mathbb{R}/\mathbb{Z}$ be the quotient space where the integers $\mathbb{Z} \subset \mathbb{R}$ are identified to a single point. Prove that X is connected, Hausdorff, and non-compact.

(4) Let (X, d_X), (Y, d_Y) be metric spaces. Suppose that Y is complete and $A \subset X$ is dense. Let $f\colon A \to Y$ be a continuous function (where A is equipped with the subspace topology).

  (a) Show that if f is *uniformly continuous*, then there exists a unique continuous map $g\colon X \to Y$ with $g|_A = f$.

  (b) Show that such a g need not exist without the assumption of uniform continuity.

(5) Prove that if a compact connected Hausdorff space is countable, then it has exactly one point.

(6) A space is called *totally disconnected* if the only connected subsets are single points.

  (a) Suppose $\{X_\alpha\}_{\alpha \in A}$ is a family of totally disconnected sets. Show that

$$X_A := \prod_{\alpha \in A} X_\alpha,$$

  equipped with the product topology, is totally disconnected.

  (b) Let A be a countably infinite set, and let $X_\alpha \simeq \{0, 1\}$ (a two-point space with the discrete topology) for each $\alpha \in A$. Show that X_A is not a discrete space.

(7) Let X be a topological space, and let $F\colon \mathbb{R} \times X \to \mathbb{R}$ be a continuous function. For $t \in \mathbb{R}$, define $f_t\colon X \to \mathbb{R}$ by $f_t(x) = F(x, t)$.

  (a) If X is compact, show that $f_{1/n}$ converges uniformly to f_0.

  (b) Show by example that this need not be true if X is not compact.