• Each problem is worth 10 points.
• Passing Standard: Do 8 of the following 10 problems and
  – Master’s level: 45 points with 3 questions essentially complete
  – Ph. D. level: 55 points with 4 questions essentially complete
• Justify your reasoning!

1. (a) Write down Cauchy–Riemann equations in polar coordinates.
(b) Use part (a) to show that the main branch of \( \text{Log} \) is a holomorphic function. Here we define \( \text{Log}(re^{i\theta}) := \ln(r) + i\theta \) for \( r > 0, -\pi < \theta < \pi \).

2. Prove the Schwarz reflection principle. Namely, let \( \Omega \subset \mathbb{C} \) be an open set symmetric under complex conjugation. Let

\[
\Omega^+ = \Omega \cap \{ z \mid \text{Im}(z) > 0 \}, \quad \Omega^- = \Omega \cap \{ z \mid \text{Im}(z) < 0 \}, \quad I = \Omega \cap \mathbb{R}.
\]

Suppose \( f(z) \) is a holomorphic function in \( \Omega^+ \) which extends continuously to \( \Omega^+ \cup I \). Then \( f(z) \) can be extended to a holomorphic function in \( \Omega \).

3. Find a holomorphic bijection between the region

\[
\{ z : |z| < 2, \text{Im}(z) > 1 \}
\]

and the region

\[
\{ z : |z| < 2, \text{Im}(z) < 1 \}.
\]

4. (a) Determine the number of zeroes of

\[
z^5 - z^4 + 2z^3 - 3z^2 - 5
\]

in the disk \( \{ z : |z| < 3 \} \).
(b) Evaluate the integral \( \int_C \frac{z^4 - 2z^2 + z - 3}{z^5 - z^4 + 2z^3 - 3z^2 - 5} \, dz \), where \( C \) is the positively-oriented boundary of the disc from part (a).

5. Evaluate the integral

\[
\int_0^\infty \frac{x^{1/3}}{x^2 + 9x + 8} \, dx
\]

Justify all your steps.

6. Let \( z_0 \) be an isolated singularity of an analytic function \( f \). Prove that if \( \text{Re}(f) \) is bounded from above, then \( z_0 \) is a removable singularity.

7. For each of the following functions, find all isolated singular points, classify them (into removable singularities, poles, essential singularities), and find residues at all isolated singular points:

(a) \( z^2 e^{\frac{z}{z^4 + 1}} \);  (b) \( \cot^2(z) \);  (c) \( \frac{z^{35}}{1 - z^{16}} \).
8. Find all Laurent series of \( f(z) = \frac{2z}{z^2 - 4z + 3} \) centered at the origin and specify for each the largest region over which it represents the function.

9. Prove the open mapping theorem: a holomorphic non-constant function \( f : \Omega \to \mathbb{C} \) is open, i.e. \( f(U) \) is open for any open set \( U \subset \Omega \). Here \( \Omega \subset \mathbb{C} \) is a connected open set.

10. Let \( F(z, w) = w^n + c_1(z)w^{n-1} + \cdots + c_n(z)w \), where \( c_1(z), \ldots, c_n(z) \) are entire functions. Assume that the polynomial \( F(0, w) \) has a unique and simple zero \( w_0 \) in the open unit disk \( D := \{ w : |w| < 1 \} \) and \( F(0, w) \) does not vanish on the boundary \( \{ w : |w| = 1 \} \).

(a) Prove that the integral 
\[
\frac{1}{2\pi i} \int_{|w|=1} \frac{\partial F(z, w)}{\partial w} F(z, w) \, dw
\]
is constantly equal to 1, for \( z \) in some non-empty connected open neighborhood \( U \) of 0 in the complex plane.

(b) Prove that the integral 
\[
\frac{1}{2\pi i} \int_{|w|=1} w \frac{\partial F(z, w)}{\partial w} F(z, w) \, dw
\]
is a well defined holomorphic function \( \varphi(z) \) of \( z \) in some non-empty connected open neighborhood \( U \) of 0 in the complex plane. Moreover, \( \varphi(0) = w_0 \) and \( F(z, \varphi(z)) = 0 \), for all \( z \in U \).