ADVANCED CALCULUS/LINEAR ALGEBRA BASIC EXAM

UMASS AMHERST, SEPTEMBER 2015

Complete 7 of the following 9 problems. Please show your work. The passing standards are:

- Master's level: 60% with three questions essentially complete (including one from each part);
- Ph.D. level: 75% with two questions from each part essentially correct.

Linear Algebra

(1) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.
  (a) Show that there is the following containment of subspaces:
$$\mathbb{R}^n \supseteq \operatorname{Im}(T) \supseteq \operatorname{Im}(T^2) \supseteq \operatorname{Im}(T^3) \supseteq \dots .$$
  (b) Show that that for some positive integer $m \geq 1$, there is equality
$$\operatorname{Im}(T^k) = \operatorname{Im}(T^{k+1}) \text{ for all } k \geq m.$$
  (c) Let $W = \operatorname{Im}(T^m)$ for the m in part (b). Thus T maps W to W. Show that the restriction of T to the subspace W is invertible.

(2) Consider the following matrix, which is in Jordan canonical form:
$$A = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$
  (a) Write A as the sum $D + N$, where D is a diagonal matrix, N is a nilpotent matrix, and D and N commute with each other. Recall a matrix N is *nilpotent* if it satisifes $N^k = 0$ for some positive integer k.
  (b) Compute A^{2015} .
  (c) Find the Jordan canonical form of A^{2015}.

(3) Let V be the real vector space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. For $f(x), g(x) \in V$, define
$$\langle f, g \rangle := \int_{-1}^{1} f(x)g(x)dx.$$
  (a) Show that $\langle \cdot, \cdot \rangle$ defines an inner product on V. Namely, prove that the above form is bilinear, symmetric, and positive definite.

(b) Let V_3 be the subspace of V of dimension four consisting of polynomials of degree at most 3. That is,

$$V_3 := \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_i \in \mathbb{R}\}.$$

Find a basis $\{p_0, p_1, p_2, p_3\}$ of V_3 satisfying:

- $p_i(1) = 1$,
- degree$(p_i) = i$, and
- $\langle p_i, p_j \rangle = 0$ if $i \neq j$.

(4) Let A and B be two $n \times n$ complex matrices. Let $f_B(x) := \det(xI - B)$ be the characteristic polynomial of B. Show that the $n \times n$ matrix $f_B(A)$ is invertible if and only if A and B have no common eigenvalue.

Advanced Calculus

(5) Let $\mathbf{F}(x, y) = (\frac{1}{2}y^2 - y, xy)$ be a vector field in the plane. Denote by C the triangular path in the plane with vertices $(0, 0), (2, 0)$, and $(0, 4)$, traversed counterclockwise. Compute the line integral

$$\int_C \mathbf{F} \bullet \mathbf{dr}$$

in two ways:

(a) Directly, by parametrizing C.

(b) Using Green's theorem.

(6) Let $f(x, y) = 2x^2 + x + y^2 - 2$. Consider the domain $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$.

(a) Explain, in one sentence, why $f(x, y)$ has both a maximum and minimum value on D.

(b) Find the maximum and minimum values on D and the points in D where they are attained.

(7) Let $f : [-1, 1] \to \mathbf{R}$ be a continuous, one-to-one function. Show that f is either increasing or decreasing.

(8) For any positive integer m, denote as usual $m! := 1 \times 2 \times \cdots \times m$. We also define $0! := 1$.

(a) For any positive integer n, show that

$$(n - 1)! \leq n^n e^{-n} e \leq n!$$

Hint: Consider (finite) Riemann sums associated to the integral $\displaystyle\int_1^n \ln x \, dx$.

(b) Deduce that the sequence $\{a_n\}$ with

$$a_n := \frac{(n!)^{1/n}}{n}$$

converges to $1/e$.

(9) Determine whether or not the series

$$\frac{\sin(x)}{1} + \frac{\cos(2x)}{4} + \frac{\sin(3x)}{9} + \frac{\cos(4x)}{16} + \frac{\sin(5x)}{25} + \frac{\cos(6x)}{36} + \ldots$$

is uniformly convergent on $[-\pi, \pi]$. Also, determine whether or not the function defined by the series is continuous on $[-\pi, \pi]$.