Complete 7 of the following 9 problems. Please show your work. The passing standards are:

- Master’s level: 60% with three questions essentially complete (including one from each part);
- Ph.D. level: 75% with two questions from each part essentially correct.

### Linear Algebra

(1) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.

(a) Show that there is the following containment of subspaces:

$$\mathbb{R}^n \supseteq \text{Im}(T) \supseteq \text{Im}(T^2) \supseteq \text{Im}(T^3) \supseteq \ldots.$$  

(b) Show that for some positive integer $m \geq 1$, there is equality

$$\text{Im}(T^k) = \text{Im}(T^{k+1})$$

for all $k \geq m$.

(c) Let $W = \text{Im}(T^m)$ for the $m$ in part (b). Thus $T$ maps $W$ to $W$. Show that the restriction of $T$ to the subspace $W$ is invertible.

(2) Consider the following matrix, which is in Jordan canonical form:

$$A = \begin{pmatrix}
3 & 1 & 0 & 0 & 0 \\
0 & 3 & 1 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & -2 & 1 \\
0 & 0 & 0 & 0 & -2
\end{pmatrix}.$$  

(a) Write $A$ as the sum $D + N$, where $D$ is a diagonal matrix, $N$ is a nilpotent matrix, and $D$ and $N$ commute with each other. Recall a matrix $N$ is nilpotent if it satisfies $N^k = 0$ for some positive integer $k$.

(b) Compute $A^{2015}$.

(c) Find the Jordan canonical form of $A^{2015}$.

(3) Let $V$ be the real vector space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. For $f(x), g(x) \in V$, define

$$\langle f, g \rangle := \int_{-1}^{1} f(x)g(x)dx.$$  

(a) Show that $\langle \cdot, \cdot \rangle$ defines an inner product on $V$. Namely, prove that the above form is bilinear, symmetric, and positive definite.
(b) Let $V_3$ be the subspace of $V$ of dimension four consisting of polynomials of degree at most 3. That is,

$$V_3 := \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 | a_i \in \mathbb{R} \}.$$ 

Find a basis $\{ p_0, p_1, p_2, p_3 \}$ of $V_3$ satisfying:

- $p_i(1) = 1$,
- degree$(p_i) = i$, and
- $\langle p_i, p_j \rangle = 0$ if $i \neq j$.

(4) Let $A$ and $B$ be two $n \times n$ complex matrices. Let $f_B(x) := \det(xI - B)$ be the characteristic polynomial of $B$. Show that the $n \times n$ matrix $f_B(A)$ is invertible if and only if $A$ and $B$ have no common eigenvalue.

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**Advanced Calculus**

(5) Let $F(x, y) = (\frac{1}{2} y^2 - y, xy)$ be a vector field in the plane. Denote by $C$ the triangular path in the plane with vertices $(0, 0)$, $(2, 0)$, and $(0, 4)$, traversed counterclockwise. Compute the line integral

$$\int_C F \cdot dr$$

in two ways:

(a) Directly, by parametrizing $C$.

(b) Using Green’s theorem.

(6) Let $f(x, y) = 2x^2 + x + y^2 - 2$. Consider the domain $D = \{ (x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 4 \}$.

(a) Explain, in one sentence, why $f(x, y)$ has both a maximum and minimum value on $D$.

(b) Find the maximum and minimum values on $D$ and the points in $D$ where they are attained.

(7) Let $f : [-1, 1] \to \mathbb{R}$ be a continuous, one-to-one function. Show that $f$ is either increasing or decreasing.

(8) For any positive integer $m$, denote as usual $m! := 1 \times 2 \times \cdots \times m$. We also define $0! := 1$.

(a) For any positive integer $n$, show that

$$(n-1)! \leq n^n e^{-n} \leq n!$$

Hint: Consider (finite) Riemann sums associated to the integral $\int_1^n \ln x \, dx$.

(b) Deduce that the sequence $\{ a_n \}$ with

$$a_n := \frac{(n!)^{1/n}}{n}$$

converges to $1/e$.

(9) Determine whether or not the series

$$\frac{\sin(x)}{1} + \frac{\cos(2x)}{4} + \frac{\sin(3x)}{9} + \frac{\cos(4x)}{16} + \frac{\sin(5x)}{25} + \frac{\cos(6x)}{36} + \cdots$$

is uniformly convergent on $[-\pi, \pi]$. Also, determine whether or not the function defined by the series is continuous on $[-\pi, \pi]$. 