Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Topology

August 25, 2014

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.

Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

(1) Let A be a path-connected subspace of a space X and $a_0 \in A$. Show that the inclusion induces a surjection from $\pi_1(A, a_0)$ to $\pi_1(X, a_0)$ if and only if every path in X with endpoints in A is path-homotopic to a path in A.

(2) Say $p\colon X \to Y$ is quotient map. Recall that the subsets $\{p^{-1}(y)\} \subset X$, as y ranges over Y, are called the *fibers* of p. Suppose Y is connected and that the fibers of p are connected. Prove X is connected.

(3) Recall that a space is said to be *first countable* if it has a countable basis at each of its points, and is said to be *second countable* if it has a countable basis for its topology. Consider $\mathbb{R}^\omega$ equipped with the product topology and the uniform topology. Which are first countable? Which are second countable?

(4) Let $p\colon E \to B$ be a covering map. Suppose points are closed in B. Let $A \subset E$ be compact. Prove that for every $b \in B$, the intersection $A \cap p^{-1}\{b\}$ is finite.

(5) Let X be compact and Hausdorff. Let $f\colon X \to Y$ be continuous, closed, and surjective. Prove that Y is Hausdorff.

(6) Let (M, d) be a metric space and suppose K and H are subsets of M. For $x \in M$ define $d(x, K) = \inf_{y \in K} d(x, y)$ and define $d(H, K) = \inf_{x \in H} d(x, K)$.
   (a) Prove that if K is closed and H is compact, then $d(H, K) = 0$ if and only if $H \cap K \neq \emptyset$.
   (b) Show by the way of an example that if K and H are closed in M, then it is possible for $H \cap K = \emptyset$ and $d(H, K) = 0$. (Hint: Find an example where $M = \mathbb{R}^2$.)

(7) Let X be compact, Y be Hausdorff, and $f\colon X \to Y$ be a continuous bijection.
   (a) Prove that f is a homeomorphism.
   (b) Give an example to show that if X is not compact, the statement in (a) doesn't hold.