DEPARTMENT OF MATHEMATICS AND STATISTICS
UMASS - AMHERST
BASIC EXAM - STATISTICS
August 25, 2014

Work all problems. Show all work. Explain your answers. State the theorems used whenever possible. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. Consider $X_1, X_2, \dots X_n$ independent and identically distributed with uniform distribution on $(0, \theta)$.
   (a) (5 points) Find a sufficient statistic for θ.
   (b) (5 points) Find the MLE $\hat{\theta}$, for θ.
   (c) (5 points) Can you use standard asymptotic theory to show $\sqrt{n}(\hat{\theta} - \theta) \to_d N(0, \sigma^2)$, for some σ^2? Why or why not? Note: you do not need to do this computation, just state whether or not you would be able to do it, and why.

   Now consider a Bayesian treatment of this problem, with a Uniform$(0, 1)$ prior on θ.
   (d) (5 points) Find the posterior distribution of θ. Show your work. You need not find the normalizing constant exactly, but show how you would find it.
   (e) (5 points) Show how you would find the Bayes estimator for θ with squared error loss.
2. (7 points) Suppose 10000 random people in the population are tested for a disease. The test used has a $\alpha = .1$ type-I error rate and a $\beta = 0$ type-II error rate. Suppose 1% of the population has the disease. Given that a person tests positive for the disease, what is the probability that that person actually has the disease?
3. Let $X_1, X_2, \dots X_n$ be iid Exponential θ random variables, such that $E(X_i) = \theta$.
   (a) (5 points) Find the UMVUE for θ. Justify.
   (b) (5 points) Find the Cramer-Rao lower bound for the variance of an estimator of θ. Is this bound attained?
   (c) (5 points) Show that the best unbiased estimator for θ^2 is $\bar{X}^2 \left(\frac{n}{n+1}\right)$. Is this the MLE for θ^2?
   (d) (5 points) Find the Cramer-Rao lower bound for the variance of an estimator of θ^2.
4. Suppose $y_1, y_2, \dots y_n$ are an iid sample from a $N(\mu, \sigma^2)$ distribution.
   (a) (4 points) Find the log-likelihood for (μ, σ^2).
   (b) (5 points) Find the MLEs for μ and for σ^2.
   (c) (5 points) Show that the MLE for μ is unbiased and that the MLE for σ^2 is biased.
   (d) (4 points) Propose an unbiased estimator for σ^2 and show that it is unbiased. Does this estimator have larger or smaller variance than the MLE?
5. Consdier $x_1, x_2, \dots x_n$, an iid sample from a Bernoulli(p) distribution.
   (a) (4 points) Find the MLE of p.
   (b) (4 points) Find the information, $I_n(p)$.
   (c) (4 points) Find the asymptotic distribution of the MLE.
   (d) (4 points) Find a $(1 - \alpha)$ asymptotic confidence interval for p.
   (e) (4 points) Consider $\tau = p(1 - p)$. What is the MLE, $\hat{\tau}$, of τ?
   (f) (3 points) Is $\hat{\tau}$ biased or unbiased for τ?
   (g) (4 points) What is the asymptotic variance of $\hat{\tau}$?
   (h) (3 points) Is $\hat{\tau}$ consistent for τ? Why or why not?