DEPARTMENT OF MATHEMATICS AND STATISTICS
UMASS - AMHERST
BASIC EXAM - PROBABILITY
August 2014

Work all problems. Show all work. Explain your answers. State the theorems used whenever possible. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. Suppose that X is uniformly distributed on the interval (0,1) and that $r > 0$ is a constant. Suppose also that X_1 and X_2 are independent random variables, both uniformly distributed on the interval (0,1).
   1) (7 points) Find the moment-generating function for X.
   2) (7 points) If s is a fixed constant, derive the moment-generating function of $Z = rX + s$. What is the distribution of Z? Why?
   3) (7 points) Find the probability density function for $U = X_1X_2$.

2. Let X_1 and X_2 have the joint probability density function given by:
$$f(x_1, x_2) = \begin{cases} k(4-x_2), & 0 \le x_1 \le x_2 \le 4 \\ 0, & \text{otherwise} \end{cases}$$
   For the following parts, you can leave your answers in terms of integrals with explicit limits. You do not need to give the final numerical answers.
   1) (7 points) Find k. Draw a picture to show the x_1 and x_2 values where the density is non-zero.
   2) (7 points) Find the marginal density functions for X_1 and X_2. Are X_1 and X_2 independent? Why or why not?
   3) (7 points) Find $P(X_2 \ge 2.1 | X_1 = 1.4)$.

3. Consider an urn containing sixteen marbles of same size and weight of which two are red, five are yellow, three are green, and six are blue. The marbles are all mixed and then randomly one marble is picked from the urn and its color is recorded. Then this marble is returned to the urn and again all the marbles are mixed, followed by randomly picking a marble from the urn and its color recorded. Again this marble is also returned to the urn and the experiment continues in this fashion. This process of selecting marbles is called sampling with replacement. After the experiment is run n times, suppose that one looks at the number of red (X_1), yellow (X_2), green (X_3) and blue (X_4) marbles which are selected.
   1) (7 points) What is the joint distribution of (X_1, X_2, X_3, X_4)?
   2) (7 points) What is the mean and variance of X_3?
   3) (7 points) If $n = 15$, what is the conditional distribution of (X1, X2, X3) given that X_4 = 5?

4. Let X_1,, ..., X_n, be iid N(μ_1, σ^2), Y_1,, ..., Y_n, be iid N(μ_2, 3σ^2) where -∞< μ_1, μ_2< ∞, 0 < σ < ∞. Also suppose that the X′s and Y′s are independent. Denote for n ≥ 2, $\bar{X}_n = n^{-1}\sum_{i=1}^{n} X_i$, $\bar{Y}_n = n^{-1}\sum_{i=1}^{n} Y_i$, $S_{1n}^2 = (n-1)^{-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$, $S_{2n}^2 = (n-1)^{-1}\sum_{i=1}^{n}(Y_i - \bar{Y}_n)^2$, and $T_n = S_{1n}^2 + S_{2n}^2/3$.

1) (7 points) Show that $V_n = \frac{1}{2\sigma}\sqrt{n}\,(\bar{X}_n - \bar{Y}_n - \mu_1 + \mu_2)$ is distributed as $N(0,1)$;
2) (6 points) Show that $(n-1)T_n/\sigma^2$ is distributed as $\chi^2_{2(n-1)}$;
3) (6 points) Are $V_{n,}\ T_n$ independent? Why or why not?
4) (6 points) Show that $U_n = \sqrt{n}((\bar{X}_n - \bar{Y}_n - \mu_1 + \mu_2)/\sqrt{2T_n}$ is distributed as the Student's $t_{2(n-1)}$;
5) (6 points) Show that $T_n \overset{P}{\rightarrow} 2\sigma^2$ as $n \rightarrow \infty$, that is, T_n converges to $2\sigma^2$ in probability;
6) (6 points) Show that $U_n \overset{D}{\rightarrow} N(0,1)$ as $n \rightarrow \infty$, that is, U_n converges to a standard normal random variable in distribution.