DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
MASTER'S OPTION EXAM — APPLIED MATH
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Do 5 of the following questions. Each question carries the same weight. Passing level is 60% and at least two questions substantially correct.

1. [20 points] A nonlinear oscillator with displacement $x \in R^1$ is governed by the differential equation:

$$\frac{d^2x}{dt^2} + \frac{dV}{dx} = 0\,, \qquad \text{with potential} \quad V = \frac{1}{2}x^2 - \frac{1}{3}x^3\,.$$

(a) Reformulate this dynamical equation as a two-dimensional system of first-order equations. Determine the equilibrium points of the system.
(b) Analyze the stability of each equilibrium point and sketch the entire phase portrait.
(c) Find a function $H = H(x, \dot{x})$ on the phase plane that is constant on each solution trajectory.

2. [20 points] Consider the PDE: $u_x + \sin(x)u_y = 0$.
(a) Find its general solution.
(b) Plot the characteristic curves.
(c) Find the solution that satisfies: $u(0, y) = e^y$.

3. [20 points] Solve the diffusion equation in 2 dimensions with Dirichlet boundary conditions in the boundaries of the square formed between the points (0,0), (0,1), (1,0) and (1,1). Consider as initial condition $u(x, y, 0) = \phi(x, y)$ and give the answer in terms of a Fourier series whose coefficient you should evaluate.

4. [20 points] For the system

$$\dot{x} = xy - 1$$
$$\dot{y} = x - y^3$$

(a) Identify all the fixed points.
(b) Classify the fixed points.
(c) Sketch the neighboring trajectories and try to fill in the rest of the phase portrait.

5. [20 points] For the dynamical system

$$\dot{x} = rx - \sin(x)$$

(a) Identify bifurcations that arise and classify their types.
(b) Find the critical points at which they occur.
(c) Construct a full bifurcation diagram of the fixed point solutions $x^\star$ as a function of r.

6. [20 points] Consider the wave equation with a transverse elastic force

$$\rho u_{tt} - T u_{xx} + ku = 0\,, \quad -\infty < x < \infty, \quad t > 0\,,$$

with initial data

$$u(x,0) = \phi(x)\,, \quad u_t(x,0) = \psi(x)$$

and assuming that u and its derivatives vanish at $\pm\infty$. Here ρ, T, and k are positive constants.
(a) Show that the total energy

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} \rho u_t^2(x,t) + T u_x^2(x,t) + k u^2(x,t) dx\,,$$

is constant in time.
(b) (15 pts) Using (a), show that the solution to the initial value problem is unique.

7. [20 points] Consider the traffic flow equation

$$\rho_t + [q(\rho)]_x = 0\,, \quad -\infty < x < \infty, \quad t > 0\,,$$

where $\rho = \rho(x,t)$ is the vehicle density at spatial location x and at time t, while $q = q(\rho) = \rho v(\rho)$ is their flux. Furthermore, the average speed $v = v(\rho)$ is given by the constitutive relation

$$v(\rho) = v_m\big(1 - \frac{\rho}{\rho_m}\big)$$

where v_m denotes the speed limit and ρ_m is the maximum density (bumper-to-bumper traffic).

(a) Solve the PDE with initial datum

$$\rho(x,0) = \begin{cases} \rho_m\,, & x \le 0 \\ 0\,, & x > 0 \end{cases}$$

(b) Solve the PDE with initial datum

$$\rho(x,0) = \begin{cases} \frac{1}{8}\rho_m\,, & x \le 0 \\ \rho_m\,, & x > 0 \end{cases}$$

(c) Explain the practical meaning in traffic terms of the two different initial data and accordingly interpret the corresponding solutions in parts (a) and (b).