Complete 7 of the following 9 problems. Please show your work. The passing standards are:

- Master’s level: 60% with three questions essentially complete (including one from each part);
- Ph.D. level: 75% with two questions from each part essentially correct.

**Linear Algebra**

(1) Let \( V_1, V_2 \) be vector subspaces of \( \mathbb{R}^n \). Prove that
\[
\dim(V_1 \cap V_2) \geq \dim V_1 + \dim V_2 - n.
\]

(2) Let \( V \) be the vector space of all \( 3 \times 3 \) matrices with real entries, and consider the linear transformation \( T: V \to V \) given by \( T(X) = AX +XA \), where
\[
A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]
Compute the determinant \( \det T \).

(3) Let \( A \) be an \( n \times n \) complex matrix. Prove that \( A^n = 0 \) if and only if \( I_n - tA \) is invertible for all nonzero \( t \in \mathbb{C} \).

(4) Let \( V \) be the subspace of \( \mathbb{R}^4 \) generated by the vectors
\[
v_1 = (1, 2, 0, 1), \quad v_2 = (0, 1, 2, -1), \quad v_3 = (2, 0, 1, -1).
\]
Let \( W \) be the subspace generated by
\[
w_1 = (3, 1, 1, -1), \quad w_2 = (0, 2, 2, 1).
\]
Find the dimension and a basis of \( V \cap W \) and \( V + W \).
Advanced Calculus

(5) Let $S$ be the surface
\[ z = x^2 + y^2, \quad z \leq 1, \]
oriented so that the normal vector has positive $z$-coordinate, and let $\mathbf{F}$ be the vector field $(yz, -xz + \sin(z), e^{x^2+y^2})$. Compute the surface integral
\[ \int_S \mathbf{F} \cdot d\mathbf{A}. \]

(6) Find, with proof, a real number $C$ so that
\[ \left| C - \int_0^1 \frac{\sin x}{x} \, dx \right| < .01. \]

(7) Consider a vector field $\mathbf{F}$ on $\mathbb{R}^3 \setminus \{0\}$ of the form
\[ \mathbf{F} = g(||\mathbf{x}||)\mathbf{x}, \]
where $g: (0, \infty) \to \mathbb{R}$ is a $C^1$ function. Show that for any closed curve $C$ in $\mathbb{R}^3 \setminus \{0\}$, the line integral
\[ \int_C \mathbf{F} \cdot d\mathbf{s} \]
vanishes.

(8) Let \( \{f_n\} \) be a sequence of continuously differentiable functions on \([a, b]\), and suppose that $f_n \to f$ pointwise, and $f'_n \to g$ uniformly on \([a, b]\). Show that
(a) $f_n \to f$ uniformly, and
(b) $f$ is differentiable, and $f' = g$.

(9) (a) Show that, if \( \{a_n\} \) is a nonnegative decreasing sequence, the series \( \sum_{n=1}^\infty a_n \) converges if and only if
\[ \sum_{n=1}^\infty 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots \]
converges.
(b) Use part (a) to show that
\[ \sum_{n=1}^\infty \frac{1}{n^p} \quad \text{and} \quad \sum_{n=1}^\infty \frac{1}{n(\log n)^p} \]
converge if and only if $p > 1$. 