Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. 

**Passing standard:** 70% with three problems essentially complete. 

Justify all your answers.

1. Prove or Disprove:
   (a) Let $M$ be a smooth oriented manifold, $\psi_t$, $t \in \mathbb{R}$, be a smooth flow generated by a smooth vector field. Then for any $t$, $\psi_t : M \to M$ is orientation-preserving.
   
   (b) There exists an $n \geq 1$, such that the tautological line bundle $E$ over the real projective space $\mathbb{R}P^n$ is trivial. (Here by definition, the fiber of $E$ over a point $p = \mathbb{R}x \in \mathbb{R}P^n$, where $0 \neq x \in \mathbb{R}^{n+1}$, is the 1-dimensional subspace $E_p = \mathbb{R}x \subset \mathbb{R}^{n+1}$. Note that $E$ may be regarded as a sub-bundle of the trivial $\mathbb{R}^{n+1}$ bundle over $\mathbb{R}P^n$.)

2. Let $M = S^2 \times S^2$, and let $\tau$ be the free involution on $M$ which sends $(x, y)$ to $(-x, -y)$, $x, y \in S^2 \subset \mathbb{R}^3$.
   (a) Compute the De Rham cohomology groups of $M$ via the Mayer-Vietoris sequence.
   (b) Determine the induced action of $\tau$ on the De Rham cohomology groups of $M$.

3. Let $M$ be a compact smooth $n$-manifold (without boundary) and $f : M \to \mathbb{R}$ be a smooth function. Show that for any interval $[a, b] \subset \mathbb{R}$ which does not contain any critical values of $f$, $f^{-1}([a, b])$ is diffeomorphic to $f^{-1}(a) \times [a, b]$. Furthermore, suppose $n = 2$ and $f$ has only two critical points $p, q \in M$ such that there are local coordinates $x, y$ near $p, q$ in which $f$ is given by $f(x, y) = f(p) + x^2 + y^2$ and $f(x, y) = f(q) - x^2 - y^2$ respectively. Show that in this case $M$ must be homeomorphic to the 2-sphere $S^2$.

4. Let $G$ be a finite group acting smoothly on a smooth $n$-manifold $M$. For any $p \in M$, we let $G_p$ be the subgroup of $G$ defined by $G_p = \{g \in G | g \cdot p = p \}$. Show that for any $p \in M$, there exists a local chart $(U, \phi)$ centered at $p$, such that (1) $g \cdot U = U$ for all $g \in G_p$, (2) there is a linear action of $G_p$ on $\mathbb{R}^n$ such that $\phi : U \to \mathbb{R}^n$ is $G_p$-equivariant, i.e., for any $q \in U$, $g \in G_p$, $\phi(g \cdot q) = g \cdot \phi(q)$. (Hint: consider the exponential map $\exp_p$.)
5. Consider smooth vector fields $U = \frac{\partial}{\partial x} - y \frac{\partial}{\partial z}$, $V = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}$ on $\mathbb{R}^3$.

(a) Show that $[U, V] = 0$.

(b) Find a local coordinate chart $(u, v, w)$ centered at the origin such that $U = \frac{\partial}{\partial u}$ and $V = \frac{\partial}{\partial v}$.

6. Consider the Riemannian metric $g = e^{2u}(dx^2 + dy^2)$ on an open subset $M \subset \mathbb{R}^2$ where $u: M \to \mathbb{R}$ is a smooth function. Calculate the Levi-Civita connection, the curvature, and the geodesic equations for this metric. Specialize your formulas to the case when $u = -\ln y$ and $M$ is the upper half plane $y > 0$. Find at least one non-trivial geodesic in this case.

7. Let $(M, g)$ be an oriented connected Riemannian manifold.

(a) Define the Hodge Laplace operator $\triangle$ on functions $f \in C^\infty(M)$.

(b) Show that for $M$ compact the only solutions to $\triangle f = 0$ are constant functions.

(c) Show that for $M$ compact a necessary condition for the solvability of $\triangle f = h$ is $\int_M h \text{dvol}_g = 0$.

8. Let $L \to M$ be a complex line bundle over a real manifold $M$. For a connection $\nabla$ on $L$ we denote the curvature 2-form by $R^\nabla \in \Omega^2(M, \mathbb{C})$.

(a) Show that $R^\nabla$ is a closed form.

(b) Show that the DeRham cohomology class of $R^\nabla$ does not depend on the connection $\nabla$. 