Department of Mathematics and Statistics
University of Massachusetts

ADVANCED EXAM — DIFFERENTIAL EQUATIONS
AUGUST 2014

Do five of the following seven problems. All problems carry equal weight. Passing level: 75% with at least three substantially complete solutions.

1. (a) Let M be a constant matrix. State (without proof) necessary and sufficient conditions on the eigenvalues of M which guarantee that:
   i. $w(t) \to 0$ as $t \to \infty$ for every solution $w(t)$ of the system $w' = Mw$;
   ii. every solution $w(t)$ of the system $w' = Mw$ is bounded for $t \in \mathbb{R}$.

   (b) Let B be a constant matrix and consider the second-order system $u'' = Bu$. Is it possible that $u(t) \to 0$ as $t \to 0$ for every solution $u(t)$? Under what assumptions on B is every solution $u(t)$ bounded for $t \in \mathbb{R}$? Justify your answers.

2. Consider the two-dimensional dynamical system:

$$\dot{x} = -x(1 + \alpha y)\,,$$
$$\dot{y} = \beta x^2 - \gamma y\,.$$

   Assume that $0 < \alpha, \beta, \gamma < 1$.

   (a) Show that the quadrant $Q = \{(x, y) \,:\, x \geq 0,\ y \geq 0\}$ is a (positively) invariant set for this system.

   (b) Construct a Lyapunov function for this system (relative to the equilibrium point $(0, 0)$) .

   (c) For the trajectory with initial condition, $x(0) = x_0 > 0,\ y(0) = 0$, find an estimate for $\max\{\, y(t) \,:\, 0 \leq t < +\infty\}$ in terms of x_0 and the constants α, β, γ.

3. (a) State the Poincare-Bendixson theorem.
   (b) Consider the system

$$\dot{x} = -5y, \qquad \dot{y} = 5x + y(8 - 2x^2 - 3y^2).$$

   Show that all disks $B_R = \{x^2 + y^2 \leq R^2\}$ are invariant for $R \geq k$; identify k.
   (c) Show that this system admits at least one periodic orbit.

4. (a) Show that the initial value problem for the backwards heat equation

$$u_t + u_{xx} = 0 \quad \text{with} \quad u^n(x,0) = \frac{1}{n}\sin(nx)$$

   is ill-posed by considering the limit of data and solution as $n \to \infty$.
   (b) Show that the boundary value problem

$$yu_x - xu_y = 0, \quad (x,y) \in \Omega = \{x^2 + y^2 \leq a^2\}, \quad \text{with BC} \quad u|_{\partial\Omega} = x,$$

   has no solution for any $a > 0$.

5. Consider the fourth-order PDE (governing the transverse vibrations of an elastic rod),

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = f(x,t), \qquad \text{in } 0 < x < 1, \ \ t > 0,$$

together with the boundary conditions (for clamped ends),

$$u(0,t) = 0 = \frac{\partial u}{\partial x}(0,t), \quad u(1,t) = 0 = \frac{\partial u}{\partial x}(1,t),$$

and the initial conditions,

$$u(x,0) = \phi(x), \qquad \frac{\partial u}{\partial t}(x,0) = \psi(x).$$

Assume that the given data, f, ϕ, ψ are smooth and bounded functions.

   (a) Use the energy method to prove that classical solutions of this initial boundary-value problem are unique.
   HINT: Consider an integral of the form

$$E = \frac{1}{2}\int_0^1 \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx.$$

(b) Suppose that the PDE is modified as follows

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} + \omega^2 u = f(x,t)\,, \qquad \text{in } 0 < x < 1\,,\;\; t > 0\,,$$

for a constant coefficient ω. How is the uniqueness argument in part (a) adapted to apply to this modified PDE ?

6. Consider the harmonic function, $u(x,y,z)$, in the half ball

$$\Omega = \{\,(x,y,z) \in R^3 \,:\, x^2+y^2+z^2 < 1\,,\;\; z > 0\,\}\,,$$

having the boundary conditions

$$u(x,y,0) = 1 \quad \text{for } x^2+y^2 < 1\,, \qquad u(x,y,z) = 0 \quad \text{for } x^2+y^2+z^2 = 1,\; z > 0\,.$$

(a) Prove that $0 < u(0,0,z) < 1 - z$ for all $0 < z < 1$.

(b) Use a maximum principle argument to show that

$$\frac{\partial u}{\partial z}(x,y,0) \le -1 \qquad \text{for all} \quad x^2+y^2 < 1\,.$$

(c) Find a positive lower bound for $\int_\Omega |\nabla u|^2 \, dxdydz$.
HINT: Make use of the result in part (b).

7. Consider the so-called Robin boundary value problem

$$-\Delta u + \gamma u = f(x) \quad \text{in } \Omega\,,$$
$$\frac{\partial u}{\partial \mathbf{n}} + \alpha u = 0 \quad \text{on } \partial\Omega\,,$$

for a smoothly bounded domain $\Omega \subset R^n$, with outward unit normal $\mathbf{n}$ on $\partial\Omega$.

(a) Introduce the appropriate Sobolev space for weak solutions u, and explain how the weak form of the BVP is derived from the classical PDE and its boundary conditions.

(b) Given that both the coefficients α and γ are positive constants, prove that this BVP has a unique weak solution for any data $f \in L^2(\Omega)$.
[Hint: Appeal either to the Lax-Milgram theorem or to a variational principle.]