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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.
2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.
3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.
4. State explicitly all results that you use in your proofs and verify that these results apply.
5. Please write your work and answers clearly in the blank space under each question and on the blank page after each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.
2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.
3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra.

1. (a) Let X be a set of points, $\mathcal{M}$ a σ-algebra of subsets of X, and μ a mapping from $\mathcal{M}$ into $[0, \infty)$ satisfying $\mu(X) < \infty$. Prove that the following two properties are equivalent.

   (a) μ is a countably additive measure on $\mathcal{M}$.

   (b) μ is a finitely additive measure on $\mathcal{M}$ that is continuous at $\emptyset$; i.e., if $\{A_n, n \in \mathbb{N}\}$ is a sequence of sets in $\mathcal{M}$ satisfying $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$ and $\cap_{n \in \mathbb{N}} A_n = \emptyset$, then $\lim_{n \to \infty} \mu(A_n) = 0$.

2. For $x \in (0, \infty)$ consider the integral

$$F(x) = \int_{(0,\infty)} \frac{1 - \exp(-xt^2)}{t^2} dt.$$

(a) Prove that $F(x) < \infty$ for all $x \in (0, \infty)$.

(b) By calculating the derivative $F'(x)$ and using the fact that $\int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi}$, compute $F(x)$ as an explicit function of x.

Justify all your steps carefully.

3. Denote by $\mathcal{B}(\mathbb{R})$ the σ-algebra of all Borel subsets of $\mathbb{R}$. Let t be any real number and A any set in $\mathcal{B}(\mathbb{R})$.

   (a) Define

   $$A + t = \{x \in \mathbb{R} : x = a + t \text{ for some } a \in A\}.$$

   Prove that $A + t \in \mathcal{B}(\mathbb{R})$.

   (b) Let μ be a finite Borel measure on $\mathbb{R}$ that is absolutely continuous with respect to Lebesgue measure m. Prove that the function mapping $t \in (0, \infty)$ to $\mu(A + t)$ is continuous. (**Hint.** Prove this first for A an interval.)

4. (a) Let $\{f_n, n \in \mathbb{N}\}$ be a sequence of real-valued, measurable functions on $[0,1]$. Assume that for each $n \in \mathbb{N}$

$$\int_0^1 f_n^2 \, dm \le \frac{1}{n^3}.$$

Prove that as $n \to \infty$, $f_n \to 0$ a.e. on $[0,1]$.

(b) Let $\{g_n, n \in \mathbb{N}\}$ be a sequence of real-valued measurable functions on $[0,1]$. Assume that for each $n \in \mathbb{N}$

$$\int_0^1 g_n^2 \, dm \to 0 \text{ as } n \to \infty.$$

By giving a counterexample, prove that it does NOT follow that as $n \to \infty$, $g_n \to 0$ a.e. on $[0,1]$.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space, and fix finite real numbers p and q satisfying $p^{-1} + q^{-1} = 1$. Let $\{f_n, n \in \mathbb{N}\}$ be a sequence of functions in $L^p(X)$ converging in $L^p(X)$ to f, and let $\{g_n, n \in \mathbb{N}\}$ be a sequence of functions in $L^q(X)$ converging in $L^q(X)$ to g. Prove that the sequence $\{f_n g_n, n \in \mathbb{N}\}$ converges in $L^1(X)$ to fg.

6. Let $\mathcal{H}$ be a Hilbert space $\mathcal{H}$ with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Assume that A and B are closed linear subspaces of $\mathcal{H}$ satisfying $A \perp B$; i.e., for all $x \in A$ and all $y \in B$, $\langle x, y\rangle = 0$. Define $A+B$ to be the set of elements $z \in \mathcal{H}$ having the representation $z = x+y$ for some $x \in A$ and some $y \in B$.

   (a) Prove that $A \cap B = \{0\}$ and that if $z \in \mathcal{H}$ has the representation $z = x + y$ for some $x \in A$ and some $y \in B$, then this representation is unique.

   (b) By using sequences, prove that $A + B$ is closed in $\mathcal{H}$.

7. By definition, a normed vector space $\mathcal{X}$ is separable if it contains a countable subset that is dense with respect to the norm on $\mathcal{X}$. For $1 \leq p \leq \infty$, let ℓ^p denote the normed vector space of real sequences $\{x_n, n \in \mathbb{N}\}$ that equals $L^p(\mathbb{N})$ for counting measure on $\mathbb{N}$. Prove or disprove the following two statements.

   (a) ℓ^1 is separable.

   (b) ℓ^∞ is separable.

8. Let f be a function mapping a finite closed interval $[a, b]$ into $\mathbb{R}$.

(a) Define the concept that f is absolutely continuous on $[a, b]$.

(b) Define the concept that f is of bounded variation on $[a, b]$.

(c) Using the definitions in parts (a) and (b), prove that if f is absolutely continuous on $[a, b]$, then f is of bounded variation on $[a, b]$.

(d) Let $T_f[a, b]$ denote the total variation of f on $[a, b]$. Prove that if f is absolutely continuous on $[a, b]$, then

$$T_f[a, b] \leq \int_a^b |f'(x)|dx < \infty.$$