UMass Amherst Algebra Advanced Exam
Friday August 29, 2014, 10AM – 1PM.

Instructions: To pass the exam it is sufficient to solve five problems including at least one problem from each of the three parts. Show all your work and justify your answers carefully.

1. Group theory and representation theory

Q1.
(a) Let $G$ be a simple group of order 168. Determine the number of elements of $G$ of order 7.
(b) Let $G$ be a group of order 20. Suppose $G$ contains an element of order 4 and has trivial center. Describe $G$ in terms of generators and relations.

Q2. Let $G$ be a finite group and $p$ a prime dividing $|G|$. Suppose $H$ is a subgroup of $G$ of index $p$.
(a) What are the possibilities for the number of conjugate subgroups of $H$?
(b) Suppose in addition that $p$ is the smallest prime dividing $|G|$. Prove that $H$ is normal.

Q3. Let $p$ be a prime. Let $G$ be the subgroup of $GL_3(\mathbb{F}_p)$ consisting of all matrices of the form
\[
\begin{pmatrix}
1 & * & * \\
0 & 1 & * \\
0 & 0 & 1
\end{pmatrix}.
\]
Compute the number of irreducible complex representations of $G$ and their dimensions.

2. Commutative Algebra

Q4.
(a) Prove that $\mathbb{Z}[\sqrt{-2}]$ is a unique factorization domain.
(b) Prove that $\mathbb{Z}[\sqrt{-3}]$ is not a unique factorization domain.

Q5. Let $R$ be a commutative ring with 1. Let $I$ and $J$ be ideals of $R$. Prove that the $R$-module $(R/I) \otimes_R (R/J)$ is isomorphic to $R/(I+J)$.

Q6. Let $k$ be an algebraically closed field. Consider the set
\[
X = \{(t^3, t^4, t^5) \mid t \in k\} \subset k^3.
\]
(a) Compute generators for the ideal $I \subset k[x, y, z]$ of polynomials vanishing at each point of the set $X$, that is,

$$I = \{ f \in k[x, y, z] \mid f(p) = 0 \text{ for all } p \in X \}.$$ 

(b) Determine the integral closure of the quotient ring $k[x, y, z]/I$ in its field of fractions.

3. Field theory and Galois theory

Q7. Let $K$ be the splitting field of the polynomial $f(x) = x^4 - 2x^2 - 1$ over $\mathbb{Q}$. Determine the Galois group $\text{Gal}(K/\mathbb{Q})$.

Q8. Let $p$ be a prime. Let $K$ be a field of order $p^{28}$. Determine the number of elements $\gamma \in K$ such that $K = \mathbb{F}_p(\gamma)$.

Q9. Let $\alpha \in \mathbb{C}$ be a root of the polynomial $f(x) = x^3 + 4x + 2$. For $n \in \mathbb{N}$, let $\zeta_n \in \mathbb{C}$ denote a primitive $n$th root of unity. Prove that $\alpha$ is not contained in the cyclotomic field $\mathbb{Q}(\zeta_n)$ for any $n$. 