(1) Let $U, V$ be dense subsets of a space $X$. Show that if $U$ is open, then $U \cap V$ is also dense in $X$. Show by example that this may not be true if $U$ and $V$ are not open.

(2) (a) Let $X$ be a metric space, and let $A \subset X$ be a subspace. If $A$ (taken with the induced metric) is complete, show that $A$ is closed in $X$.
(b) Show that a compact metric space is complete.

(3) Let $X$ be a compact, Hausdorff space. Let $A, B$ be disjoint closed subsets of $X$. Show that there exist disjoint open subsets $U$ and $V$ containing $A$ and $B$, respectively. (Hint: first start with the case when $B = \{b\}$ is a single point)

(4) Let $X$ be the quotient of $S^2$ given by gluing two points together, and let $Y$ be the quotient of the torus $S^1 \times S^1$ obtained by collapsing the circle $S^1 \times \{a\}$ to a point. Show that $X$ and $Y$ are homeomorphic.

(5) Let $X$ be a Hausdorff space. Suppose that $\{f_\alpha\}_{\alpha \in I}$ is a family of continuous functions $X \to [0, 1]$ with the property that for any $x \in X$ and any closed set $A \subset X$ with $x \notin A$, there exists an $\alpha \in I$ such that $f_\alpha(x) = 0$ and $f_\alpha(A) = \{1\}$. Give $[0, 1]^I$ the product topology. Prove that $F: X \to [0, 1]^I$, $F(x) = (f_\alpha(x))_{\alpha \in I}$ is an embedding, i.e. it is a homeomorphism onto its image.

(6) Let $f: X \to Y$ be a continuous map between path-connected spaces. The mapping cone $M_f$ is the quotient of the disjoint union of $X \times I$ and $Y$ by the equivalence relation generated by
(a) $(x, 0) \sim (x', 0)$ for all $x, x' \in X$, and
(b) $(x, 1) \sim f(x)$ for all $x \in X$.
(it can be visualized as the space obtained by gluing a cone over $X$ to $Y$ via the map $f$). Show that there is an isomorphism
$$\pi_1(M_f) \cong \pi_1(Y)/(f_*(\pi_1(X)))$$
where $\langle S \rangle$ denotes the smallest normal subgroup of $\pi_1(M_f)$ generated by $S$. 

(turn over for last problem)
(7) Let $C(\mathbb{R})$ denote the set of all continuous functions $\mathbb{R} \to \mathbb{R}$, with the uniform topology, i.e. the topology associated to the metric

$$d(f, g) = \sup_{x \in \mathbb{R}} \min \{1, |f(x) - g(x)|\}.$$ 

Consider a function $f: \mathbb{R}^2 \to \mathbb{R}$ which is continuous in the second variable, meaning that for every $t \in \mathbb{R}$ the map $x \mapsto f(t, x)$ is continuous. That means that we can define a function $F: \mathbb{R} \to C(\mathbb{R})$ by $t \mapsto F_t$ where $F_t(x) = f(t, x)$.

If $f$ is continuous, is it necessarily true that $F$ is continuous? What about the converse? Prove each result or give a counterexample.