Department of Mathematics and Statistics
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Basic Exam: Topology

August 27, 2013

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.

Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

(1) Let U, V be dense subsets of a space X. Show that if U is open, then $U \cap V$ is also dense in X. Show by example that this may not be true if U and V are not open.

(2) (a) Let X be a metric space, and let $A \subset X$ be a subspace. If A (taken with the induced metric) is complete, show that A is closed in X.
(b) Show that a compact metric space is complete.

(3) Let X be a compact, Hausdorff space. Let A, B be disjoint closed subsets of X. Show that there exist disjoint open subsets U and V containing A and B, respectively. (Hint: first start with the case when $B = \{b\}$ is a single point)

(4) Let X be the quotient of S^2 given by gluing two points together, and let Y be the quotient of the torus $S^1 \times S^1$ obtained by collapsing the circle $S^1 \times \{a\}$ to a point. Show that X and Y are homeomorphic.

(5) Let X be a Hausdorff space. Suppose that $\{f_\alpha\}_{\alpha \in I}$ is a family of continuous functions $X \to [0,1]$ with the property that for any $x \in X$ and any closed set $A \subset X$ with $x \notin A$, there exists an $\alpha \in I$ such that $f_\alpha(x) = 0$ and $f_\alpha(A) = \{1\}$. Give $[0,1]^I$ the product topology. Prove that

$$F \colon X \to [0,1]^I, \quad F(x) = (f_\alpha(x))_{\alpha \in I}$$

is an embedding, i.e. it is a homeomorphism onto its image.

(6) Let $f \colon X \to Y$ be a continuous map between path-connected spaces. The *mapping cone* M_f is the quotient of the disjoint union of $X \times I$ and Y by the equivalence relation generated by
(a) $(x, 0) \sim (x', 0)$ for all $x, x' \in X$, and
(b) $(x, 1) \sim f(x)$ for all $x \in X$.
(it can be visualized as the space obtained by gluing a cone over X to Y via the map f). Show that there is an isomorphism

$$\pi_1(M_f) \cong \pi_1(Y)/\langle f_*(\pi_1(X))\rangle,$$

where $\langle S\rangle$ denotes the smallest normal subgroup of $\pi_1(M_f)$ generated by S.

(turn over for last problem)

(7) Let $\mathcal{C}(\mathbb{R})$ denote the set of all continuous functions $\mathbb{R} \to \mathbb{R}$, with the uniform topology, i.e. the topology associated to the metric

$$d(f, g) = \sup_{x \in \mathbb{R}} \min\{1, |f(x) - g(x)|\}.$$

Consider a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ which is continuous in the second variable, meaning that for every $t \in \mathbb{R}$ the map $x \mapsto f(t, x)$ is continuous. That means that we can define a function $F\colon \mathbb{R} \to \mathcal{C}(\mathbb{R})$ by $t \mapsto F_t$ where $F_t(x) = f(t, x)$.

If f is continuous, is it necessarily true that F is continuous? What about the converse? Prove each result or give a counterexample.