DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERIC ANALYSIS EXAM
AUGUST 2013

Do five of the following problems. All problems carry equal weight.
Passing level:

Masters: 60% with at least two substantially correcct.
PhD: 75% with at least three substantially correct.

1. Determine the values of α for which the matrix

$$A = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 10 & 23 \\ 4 & 23 & \alpha \end{bmatrix}$$

is positive definite. Explain your method.

2. Using the explicit Euler method to write the discretization of the system

$$\begin{aligned} \dot{y} &= z \\ \dot{z} &= -\omega^2 y - 2z. \end{aligned}$$

For what values of ω and h is it stable?

3. Consider the equation

$$\dot{y} = f(x, y).$$

Find a, b_1, and b_2 so that the discretization

$$y_{n+1} = a y_n + h(b_1 f(x_{n-1}, y_{n-1}) + b_2 f(x_{n-2}, y_{n-2}))$$

is of maximal order.

4. Find a, b, c so that the quadrature rule

$$I(f) = af(1/2) + bf(1/4) + cf(1/8)$$

has maximal order for the integral $\displaystyle\int_0^1 f(x)\,dx$.

5. Let $x_0 < x_1 < \ldots < x_n$ be $n+1$ distinct points.

   (a) Prove that there is a unique polynomial of degree at most n that interpolates the function $f(x)$ at these nodes.

   (b) Derive the Lagrange form of the interpolation polynomial.

   (c) Find the lowest order polynomial $p(x)$ that satisfies the conditions:

$$p(0) = 3, \qquad p(1) = 4, \qquad p'(0) = -1, \qquad p'(1) = 3.$$

6. Given a vector norm $\|\cdot\|$ for the space $\mathbb{R}^n$, the induced matrix norm for an n-by-n matrix A is defined as

$$\|A\| = \max_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|}.$$

For a non-singular real matrix A,

(a) Show that the condition number $\kappa(A) = \|A\| \cdot \|A^{-1}\| \geq 1$.

(b) Find $\kappa(A)$ for orthogonal matrix A, when the Euclidean norm is used.

(c) Consider the linear system $Ax = b$ and its perturbed version $(A + \delta A)x = b + \delta b$. Show that

$$\frac{\|\delta b\|}{\|b\|} \leq \kappa(A)\frac{\|\delta A\|}{\|A\|}.$$

7. Given a fixed point iteration

$$x_{n+1} = \phi(x_n), \qquad n = 0, 1, 2, \ldots$$

where $\phi(x) = Ax + Bx^2 + Cx^3$.

(a) For $\alpha > 0$, find the constants A, B, C such that the iteration converges locally to $\dfrac{1}{\alpha}$ with order 3.

(b) Determine the maximal possible interval in which the initial guess x_0 can lie in order to ensure the convergence of the above iteration.