Complex analysis qualifying exam

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Do 8 out of the following 10 questions.
Each question is worth 10 points. To pass at the Master’s level it is sufficient to have 45 points with 3 questions essentially correct. To pass at the PhD level it is sufficient to have 55 points with 4 questions essentially correct.

Note: All answers should be justified carefully.

(1) (10 points) Let \( f : \mathbb{C} \to \mathbb{C} \) be a holomorphic function such that \( \frac{f(z)}{z} \to 0 \) as \( |z| \to \infty \). Prove that \( f \) is constant.

(2) (a) (2 points) State Rouché’s theorem.
(b) (8 points) Consider the function
\[
 f : \mathbb{C} \to \mathbb{C}, \quad f(z) = z^5 + e^z + 4.
\]
Let
\[
 \Omega = \{ z = x + iy \in \mathbb{C} \mid x < 0 \} \subset \mathbb{C},
\]
the left half plane. Show that \( f \) has exactly 3 zeroes in \( \Omega \) (counting multiplicities).

(3) (a) (5 points) Let
\[
 \Omega_1 = \{ z = x + iy \in \mathbb{C} \mid 0 < y < 1 \},
\]
a horizontal strip, and
\[
 \Omega_2 = \{ z = x + iy \in \mathbb{C} \mid x > 0 \text{ and } y > 0 \},
\]
the positive quadrant. Find a holomorphic bijection \( f : \Omega_1 \to \Omega_2 \).
(b) (5 points) Let \( \Omega_3 = \{ z \in \mathbb{C} \mid |z - 1| < 1 \text{ and } |z - i| < 1 \} \)
(a “lune”). Find a holomorphic bijection \( g : \Omega_3 \to \Omega_2 \).

(4) (a) (2 points) Let \( \Omega \subset \mathbb{C} \) be an open set, \( a \in \Omega \) a point, and
\[
f : \Omega \setminus \{a\} \to \mathbb{C}
\]
a holomorphic function. Define the residue of \( f \) at \( a \).
(b) Let \( \gamma \) denote the circle with center the origin and radius 3, traversed once counterclockwise. Compute the following contour integrals.

i. (4 points)
\[
\int_{\gamma} \frac{z^2}{(z - 2)(z + 1)^2} dz.
\]

ii. (4 points)
\[
\int_{\gamma} \frac{e^z}{\sin z} dz.
\]

(5) (10 points) Compute the improper integral
\[
\int_{-\infty}^{\infty} \frac{1}{x^6 + 1} dx.
\]

(6) Let \( f \) be a one-to-one holomorphic map from a region \( \Omega_1 \) onto a region \( \Omega_2 \). Assume that the closure of the disc \( D := \{ z : |z - z_0| < \epsilon \} \) is contained in \( \Omega_1 \). Prove that the inverse function \( f^{-1} : f(D) \to D \) is given by the integral formula
\[
f^{-1}(\omega) = \frac{1}{2\pi i} \int_{|z - z_0| = \epsilon} \frac{f'(z)}{f(z) - \omega} \cdot z dz.
\]

(7) Let \( \Omega \) be a connected open subset of the complex plane and \( f_n(z), n \geq 1 \), a sequence of holomorphic and nowhere vanishing functions on \( \Omega \). Assume that the sequence \( f_n(z) \) converges to a function \( f(z) \), uniformly on every compact subset of \( \Omega \). Prove that \( f \) is either identically zero, or never equal to zero in \( \Omega \).
(8) Let \( f(z) = a_0 + a_1z + \cdots + a_nz^n \) be a polynomial of degree \( n > 0 \). Prove that
\[
\frac{1}{2\pi i} \int_C z^{n-1} |f(z)|^2 \, dz = a_0 \bar{a}_n R^{2n},
\]
where \( C \) is the circle \(|z| = R\) traversed once counterclockwise.

(9) Let \( C \) be the circle \(|z| = 2\) traversed counter-clockwise. Compute
\[
\int_C \frac{z^{2n} \cos(1/z)}{1 - z^n} \, dz \quad \text{for all integers } n \geq 2.
\]

(10) Prove or disprove the following statements.

(a) Let \( U \) be a simply connected open subset of the complex plane. For any two points \( p, q \) in \( U \) there exists a one-to-one holomorphic map from \( U \) onto itself such that \( f(p) = q \).

(b) For any open subset \( W \) of the complex plane, any harmonic function on \( W \) is the real part of a holomorphic function on \( W \).

(c) If \( f \) and \( g \) are meromorphic on the complex plane, then so is the composition \( f \circ g \).