DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MASTER'S OPTION EXAM-APPLIED MATHEMATICS
August 2013

Do five of the following problems. All problems carry equal weight. Passing level: 60% with at least two substantially correct.

1. Consider the initial value problem

$$\frac{du}{dt} = f(u,t)$$
$$u(0) = u_0.$$

   (a) State a general existence theorem for initial value problems of the above form.

   (b) Find 2 explicit solutions of the above problem when $f(u,t) = u^{1/2}$ and $u_0 = 0$.

   (c) Explain the relationship between the number of solutions of the above problem and the hypotheses and conclusions of the above theorem.

2. Consider the first order dynamical system:

$$\dot{x} = \mu x + x^3 - x^5$$

   (a) Identify all the fixed points and the parametric regimes in which they arise.

   (b) Identify all the bifurcations and the parametric values for which they occur.

   (c) Sketch the bifurcation diagram of the fixed points $x^\star = x^\star(\mu)$ (including the stability information of the fixed points as solid lines for stable and dashed lines for unstable ones).

3. Consider a dynamical system on the plane.

   (a) Write down the 4 conditions under which a closed orbit exists based on the Poincaré-Bendixson theorem.

(b) Consider the specific example

$$\dot{r} = r(1 - r^2) + \mu r \sin(\theta)$$

$$\dot{\theta} = 1$$

Find the limit cycle that the system possesses at $\mu = 0$ and discuss its stability.

(c) Illustrate that for $0 < \mu \ll 1$, this dynamical system in an appropriate subset of the plane still satisfies the conditions of the Poincaré-Bendixson theorem and hence the limit cycle persists.

4. (a) Prove the uniqueness of the solution for the problem $u_t = ku_{xx}$, $u(x,0) = \phi(x)$, $u(0,t) = u(l,t) = 0$, by using the maximum principle.

   (b) Prove the uniqueness of the solution for the problem $u_{tt} = c^2 u_{xx}$, $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$, $u_x(0,t) = 0$, $u_x(l,t) = 0$ in $(0,l)$, by means of the energy method.

   Notice: you should *define* the energy and *prove* that it is conserved.

5. Find the most general possible solution of the PDE:

$$u_{xx} - 5u_{tt} - 4u_{xt} = 0$$

Show all your work in obtaining this solution.

6. Consider the system

$$\dot{x} = x^2 + y^2 - 2, \;\; \dot{y} = x^2 - y^2.$$

   (a) Find the equilibrium points.

   (b) Evaluate the Jacobian at each equilibrium point and find its eigenvalues.

   (c) State the nature of each equilibrium point for which it is possible to do so.

(d) Sketch the phase portrait.

7. Find the solution to the given boundary value problem

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad 0 < y < \pi, \\
\frac{\partial u}{\partial x}(0, y) = \frac{\partial u}{\partial x}(\pi, y) = 0, \quad 0 \le y \le \pi, \\
u(x, 0) = \cos x - 2\cos 4x, \quad 0 \le x \le \pi, \\
u(x, \pi) = 0, \quad 0 \le x \le \pi.
\end{aligned}$$