Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. Passing standard: 70% with three problems essentially complete. Justify all your answers.

1. Prove or disprove the following statements:
   (a) The tangent bundle \( T(RP^2 \times S^1) \) is trivial.
   (b) The connected sum of \( RP^3 \) with itself is orientable.

2. Consider the smooth map \( f : \mathbb{R}^3 \to \mathbb{R}^4 \) given by \( f(x, y, z) = (x^2 - y^2, xy, xz, yz) \).
   Let \( M \) be the image of the restriction of \( f \) on the unit sphere \( x^2 + y^2 + z^2 = 1 \).
   Show that \( M \) is an embedded submanifold of \( \mathbb{R}^4 \).

3. Use the Mayer-Vietoris sequence and induction to compute the de Rham cohomology groups of the complex projective spaces \( CP^n \).

4. Let \( D \) be the 2-dimensional smooth distribution on \( \mathbb{R}^3 \) spanned by vector fields
   \[ X = \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}, \quad Y = x \frac{\partial}{\partial x} + \frac{\partial}{\partial y} - (x^2 + y) \frac{\partial}{\partial z}. \]
   (a) Show that \( D \) is involutive.
   (b) Describe the integral submanifolds of \( D \) in \( \mathbb{R}^3 \).

5. Consider the set \( E \) over the real projective space \( RP^n \) given by
   \[ E := \bigsqcup_{x \in RP^n} E_x \]
   where for each point \( x = [x_0 : x_1 : \cdots : x_n] \in RP^n \), \( E_x \) is the unique line through the point \((x_0, x_1, \cdots, x_n)\) and the origin in \( \mathbb{R}^{n+1} \).
   (a) Show that \( E \) is naturally a smooth vector bundle over \( RP^n \).
   (b) Show that \( E \) is not isomorphic to the product bundle (i.e. the trivial bundle) over \( RP^n \) for any \( n \geq 1 \).

6. Let \( n > 0 \). Suppose \( f : M \to S^n \) is an immersion from a compact closed, connected \( n \)-manifold \( M \) to the \( n \)-sphere \( S^n \). Prove that \( f \) is a diffeomorphism.
7. Consider the noncompact surface \( S = \{(x, y, z) : z = x^2 + y^2\} \subset \mathbb{R}^3 \).

(a) Find the supremum for the Gauss curvature and the subset of \( S \) on which it is attained.

(b) Does the Gauss curvature attain its infimum on \( S \)? (Explain why or why not!)

8. Prove that the set of upper triangular real \( 3 \times 3 \) matrices with determinant 1 is a Lie group. Furthermore,

(a) How many connected components does this group have?

(b) Determine its Lie algebra and compute its dimension.