Department of Mathematics and Statistics
University of Massachusetts Amherst

Advanced Exam – Algebra. August 28, 10:00-13:00, LGRT 1322, 2013.

Passing Standard: It is sufficient to do five problems correctly, including at least one problem from each of the three parts.

1. Group Theory and Representation Theory

1. Let us fix m and n. Show that there are only finitely many conjugacy classes of finite subgroups inside $GL(n, \mathbb{C})$ of order m.

2. Show that there are no simple group of order 80.

3. Let p be a prime and G a non-abelian group of order p^3.

- **a.** Show that the center $Z(G)$ of G and the commutator subgroup of G are equal and have order p.
- **b.** Show that $G/Z(G) \cong \mathbf{Z}/p \times \mathbf{Z}/p$.

2. Commutative Algebra

4. Let $x_1(t), \ldots, x_n(t) \in \mathbb{C}[t]$ such that $A := \mathbb{C}[x_1(t), \ldots, x_n(t)] \subset \mathbb{C}[t]$ is finite over $\mathbb{C}[t]$ and $\mathbb{C}(t) = \mathbb{C}(x_1(t), \ldots, x_n(t))$. Show that $\mathbb{C}[t]$ is the normalization of A.

5. Let K be a finite field and $f \in K[x_1, \ldots, x_n] \setminus K$. Prove that there exists $y_2, \ldots, y_n \in K[x_1, \ldots, x_n]$ such that $K[y_2, \ldots, y_n] \subset K[x_1, \ldots, x_n]/(f)$ is finite. For any sufficiently large e one can choose $y_i = x_i - x_1^{e^i}$.

6. Let R be a commutative ring such that not every ideal is a prinicipal ideal.

- **a.** Show that there is an ideal I maximal with respect to the property that I is not a principal ideal.
- **b.** If I is an ideal as in (a), show that R/I is a principal ideal ring.

3. Field Theory and Galois Theory

7. Show that for the polynomial over the field F of characteristic $p > 0$ polynomial $f(x) = x^p - x - a$ is either irreducible or factors into linear factors.

8. Let L/K be a Galois extension and set $G = \mathrm{Gal}(L/K)$. Let $f(x) \in K[x]$ be a monic polynomial that splits over L and let $S \subseteq L$ be the set of roots of $f(x)$. Prove that $f(x)$ is a power of an irreducible polynomial in $K[x]$ if and only if G acts transitively on S.

9. Let $\alpha \in L$ be a primitive element for the Galois extension L/K. Suppose that there is $\sigma \in \mathrm{Gal}(L/K)$ such that $\sigma(\alpha) = \alpha^{-1}$. Prove that $[L : K]$ is even and that $[K(\alpha + \alpha^{-1}) : K] = \frac{1}{2}[L : K]$.