DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM – STATISTICS
AUGUST 2012

Work all problems. Point values are as indicated. Sixty points are needed to pass at the Master's level and seventy-five at the Ph.D. level.

1. Let $X_1,\ldots,X_n$ be iid $N(\theta,1)$. Show that

   a) (7 points) The best unbiased estimator for θ^2 is $\bar{X}^2 - 1/n$.
   b) (7 points) Calculate the variance of the best unbiased estimator.
   c) (8 points) Show that the variance is greater than the Cramer-Rao lower bound {*hint: the fourth moment of* $N(0,1)$ *is 3*}.

2. Let $X_1,\ldots,X_n$ be iid Uniform$(0,\theta)$ and let $Y = \max\{X_1,\ldots,X_n\}$. We want to test

   $$H_0: \theta = 1/2 \textit{ versus } H_1: \theta > 1/2.$$

   The Wald test is not appropriate since Y does not converge to a Normal. Suppose we decide to test this hypothesis by rejecting H_0 when $Y > c$.

   a) (7 points) Find the power function.
   b) (7 points) What choice of c will make the size of the test .05?
   c) (8 points) In a sample of size $n = 20$ with $Y = 0.48$, what is the p-value? What conclusion about H_0 would you make?

3. Let $X_1, \ldots, X_n$ be a random sample from the Geometric distribution with the following probability function:

   $$f(x \mid \lambda) = \lambda(1-\lambda)^x, \quad x = 0,1,2,\ldots; \ 0 \le \lambda \le 1.$$

   Let the prior distribution for the parameter λ be the Beta distribution with the following probability density function. (i.e. $\lambda \sim$ Beta(α,β)):

   $$f(\lambda \mid \alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\lambda^{\alpha-1}(1-\lambda)^{\beta-1};$$

   for $0 \le \lambda \le 1$, $\alpha > 0$; $\beta > 0$.

   Note that for $\lambda \sim$ Beta(α,β), the following are known:

   $$E(\lambda) = \frac{\alpha}{\alpha+\beta},$$

   $$Var(\lambda) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

a) (7 points) Find the posterior distribution of λ.
b) (7 points) Find thoue posterior mean of λ.
c) (7 points) Describe how to construct a 95% equal-tail posterior interval for λ.

4. 100 dairy calves were classified according to whether or not they caught pneumonia within 60 days of birth. Calves that caught pneumonia were further classified according to whether or not they had a second episode of pneumonia within 60 days of the first infection. The goal was to test the hypothesis, H_0, that the probability of an initial infection was equal to the (conditional) probability of a subsequent infection among animals infected initially. The data were as follows:

	Secondary Yes	Secondary No
Primary Yes	$n_{11} = 23$	$n_{10} = 31$
Primary No	---	$n_{00} = 46$

a) (7 points) Let p_1 denote the probability of a primary infection and let $p_{2|1}$ denote the conditional probability of a secondary infection given a primary infection. Let Y_i denote the classification of the ith calf, i.e., Y_i is either primary Yes and secondary Yes, primary Yes and secondary No, or primary No and secondary No. Write out the joint distribution of $Y_1, \ldots, Y_{100}$ in terms of p_1, $p_{2|1}$, n_{11}, n_{10}, and n_{00}. Give the name of the joint distribution.

b) (7 points) Under H_0, $p_1 = p_{2|1} = \pi$ (say). What is the maximum likelihood estimate of the common value π? [Please give a numerical answer, although this need not be given in decimal form.]

c) (7 points) What is the asymptotic sampling distribution of this estimate under H_0?

d) (7 points) Propose an appropriate test statistic for testing H_0. How would you determine whether there is sufficient evidence to reject the hypothesis H_0 (include explicit distribution) [No numerical answer necessary.]

e) (7 points) State clearly all the limit theorems that you have used in providing answers to (c) and (d).