DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERIC ANALYSIS EXAM
AUGUST 2012

Do five of the following problems. All problems carry equal weight.
Passing level:

Masters: 60% with at least two substantially correcct
PhD: 75% with at least three substantially correct.

1. Derive the numerical differentiation formula of the form

$$f'(x) \approx S_h f = af(x) + bf(x+h) + cf(x+2h)$$

with the high possible degree of precision

(a) Bound the error in the approximation, assuming f is smooth enough.

(b) Use Richardson extrapolation to combine S_h and $S_{h/2}$ to produce a more accurate approximation.

2. Find the linear least squares approximate p_1 to $f(t) = e^t$ on $[0, 1]$, that is, the polynomial of degree 1 for which

$$\int_0^1 [p_1(t) - f(t)]^2 dt = minumum$$

3. Consider the *natural* cubic spline function $s(x)$ defined on $[0, 2]$ which interpolates the function $f(x)$ using the following data:

x_i	0	1	2
$f(x_i)$	f_0	f_1	f_2

Therefore, $s(x)$ is defined piecewise by 2 cubic polynomials,

$$s(x) = \begin{cases} s_0(x) & 0 \le x \le 1, \\ s_1(x) & 1 \le x \le 2. \end{cases}$$

Suppose that $s_0(x) = 1 + 2x + 4x^3$. Then

(a) What are the values of f_0 and f_1?

(b) What is the value of f_2?

(c) Suppose further that $f \in C^2[0, 2]$. Show

$$\int_0^2 [s(x)'']^2\, dx \le \int_0^2 [f(x)'']^2\, dx.$$

4. Let f be an arbitrary(continuous) function on $[0,1]$, satisfying $f(x)+f(1-x)=1$ for $0 \le x \le 1$.

   (a) Show that $\int_0^1 f(x)dx = \frac{1}{2}$.

   (b) Show that the composite trapezoidal rule for computing $\int_0^1 f(x)dx$ is exact.

5. Consider the ODE initial-value problem

$$\frac{dy}{dx}(x) = f(x, y(x)),$$

   with initial data $y(x_0) = y_0$. We would like to solve this initial value problem at points $x_n = nh, n = 0, ..., N$ where $h = x_n - x_{n-1}$ for all n. Find the highest order method in the class

$$y_{n+1} = y_n + h[b_1 f(x_n, y_n) + b_2 f(x_{n-1}, y_{n-1})].$$

   i.e., find b_1 and b_2 for the above method which gives the highest order local truncation error. State the order of the method obtained.

6. (a) Let A be an $n \times n$ matrix and $\|\cdot\|_1$ denote the vector norm on R^n given by $\|v\|_1 = \sum_{i=1}^n |v_i|$. Show that the matrix induced norm from the given vector norm is given by

$$\|A\|_1 = \max_j \sum_{i=1}^n |A_{i,j}|.$$

   (b) Let $\|\cdot\|_2$ denote the norm on R^n given by $\|v\|_2 = (\sum_{i=1}^n |v_i|^2)^{1/2}$. For

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$$

   compute $\|A\|_2$.

7. Let $u, v \in \mathbb{R}^n$ be column vectors, and consider the *rank one perturbation of the identity* defined by $A = I - uv^T$.

   (a) Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha uv^T$ for some scalar α. Give an expression for α.

   (b) For what u and v is A singular? Show if A is singular, then it is a projector.

   (c) For what u and v is A an orthogonal projector?