DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
MASTER'S OPTION EXAM — APPLIED MATH
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Do 5 of the following questions. Each question carries the same weight. Passing level is 60% and at least two questions substantially correct.

1. The population $P(t)$ of a species of fish follows a logistic model, while a constant amount C of fish is harvested. The resulting model is

$$\frac{dP}{dt} = kP\Big(1 - \frac{P}{N}\Big) - C$$

where the parameters k, N, C are positive.

(a) Find the equilibrium points and precisely determine how many there are depending on the parameters k, N, C.

(b) Using the information in (a) and the usual stability analysis, draw the corresponding bifurcation diagram. Are there any regimes were the effect on the total population of increased or decreased fishing policies can be "dramatic"?

2. Consider the system

$$x' = x + y + ax(x^2 + y^2)\,, \quad y' = -x + y + ay(x^2 + y^2)\,. \tag{1}$$

(a) Set up the linearized system around the equilibrium point $(0, 0)$ and describe its behavior.

(b) How does the original nonlinear system (1) behaves as $t \to \infty$ and for different choices of the constant a, and how do you compare your result to part (a) above.

3. Consider the system

$$x' = y\,, \quad y' = -by + x - x^3\,, \tag{1}$$

and consider the quantity $E(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$.

(a) Calculate $\frac{d}{dt}E(x,y)$ on the teajectories of the system; when (for what value(s) of b) is the system conservative and when dissipative?

(b) Draw the phase portrait of the system in the conservative case and justify carefully your answer; based on the phase plane, sketch all typical "interesting" solutions $x = x(t)$ and $y = y(t)$ as functions of the t variable.

(c) Do the same in the dissipative case.

4. Assume $u = u(x)$ is a smooth solution of the Dirichlet boundary value problem for Laplace's equation on a bounded domain D with smooth boundary:

$$\Delta u = 0 \quad \text{in} \quad D\,, \quad u = h \quad \text{on} \quad \partial D\,,$$

where $h = h(x)$ is a given function. Next, we define the potential energy

$$E[w] = \frac{1}{2}\int_D |\nabla w(x)|^2 dx\,,$$

for any smooth enough function $w = w(x)$ that satisfies $w = h$ on ∂D. Show that for any such w we have that $u = u(x)$ is a minimizer of the potential energy, i.e.

$$E[w] \geq E[u]\,.$$

5. Consider the "telegraph" equation

$$\rho u_{tt} - T u_{xx} + k u_t = 0\,, \quad -\infty < x < \infty, \quad t > 0\,,$$

with initial data

$$u(x,0) = \phi(x)\,, \quad u_t(x,0) = \psi(x)$$

and assuming that u and its derivatives vanish at $\pm\infty$. Here ρ, T, and k are positive constants.

(a) Show that the total energy

$$E(t) = \frac{1}{2}\int_{\infty}^{\infty} \rho u_t^2(x,t) + T u_x^2(x,t) dx\,,$$

is decreasing in time.

(b) Using (a), show that the solution to the initial value problem is unique.

6. Consider the equation

$$u_t - k u_{xx} + a u = 0\,, \quad 0 < x < l, \quad t > 0\,,$$

with initial data

$$u(x,0) = \phi(x)\,,$$

and boundary conditions

$$u(0,t) = 0 = u(l,t)\,,$$

where k and a are constants and k is positive.
(a) Write the series solution for this problem.
(b) Can you conjecture from the formula obtained in (a) the behavior of the solution $u = u(x,t)$ for large values of t?

7. Consider the Burgers' equation for $u(x,t)$,

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = 0, \qquad (\,x \in R^1,\ t > 0\,)\,.$$

(a) Using the method of characteristics, solve the equation for the initial data

$$u_1(x,0) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ x & \text{if} \quad 0 < x \leq 1 \\ 1 & \text{if} \quad x > 0 \end{cases}$$

and

$$u_2(x,0) = \begin{cases} 1 & \text{if} \quad x \leq 0 \\ 1 - x & \text{if} \quad 0 < x \leq 1 \\ 0 & \text{if} \quad x > 0 \end{cases}$$

(b) Do solutions exist globally in time in both cases? Explain your answer and plot solutions for suitably selected typical times.