UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
ADVANCED EXAM - Probability and “Mathematical Statistics”
August 31, 2012

70 Points are required to pass with at least 25 from problems 1 and 2 and 25 from problems 3 - 5.

1. (20 PTS) Let

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left( \begin{bmatrix} \mu \\ \mu \end{bmatrix}, \Sigma \right),$$

where $\Sigma = (1-\rho)I_2 + \rho J_2$, where $J_2$ is a $2 \times 2$ matrix with all elements equal to 1, $I_2$ is that $2 \times 2$ identity matrix and $|\rho| < 1$. Let $Q_1 = (X_1 - X_2)^2$ and $Q_2 = (X_1 + X_2)^2$.

Show that

(a) $Q_1/2(1-\rho)$ has a chi-square distribution; give the degrees of freedom and non-centrality parameter (which may possibly be 0).

(b) $Q_1$ and $Q_2$ are distributed independently.

State what results you are employing to derive your answers.

2. (25 PTS) Let $\mathbf{X} = (X_1, X_2, X_3)' \sim N_3(\mathbf{0}, \mathbf{\Sigma})$, where

$$\mathbf{\Sigma} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

(a) Give the name and its associated parameters of the conditional distribution of $(X_1|X_2 = x_2, X_3 = x_3)$. Here, you can just state the result you are using, no need to derive it.

(b) Find the mean and variance of $Y = 4X_1 - 6X_2 + X_3 - 18$. You can leave the answers in terms of multiplications involving vectors and/or matrices but with numbers involved throughout.

(c) Find the distribution of Y as follows

i. **Derive** the moment generating function of $\mathbf{X}$. (If you can't derive it, at least state it).

ii. **Derive** the moment generating function of a linear combination $Y = \mathbf{c}'\mathbf{X}$, where $\mathbf{c}$ is a vector of constants (you can use the MGF from i) in doing this) and then use this to state what the distribution of $Y$ is.

3. (23 PTS) Let $\bar{X}$ and $S^2$ be the sample mean and sample variance of $n$ i.i.d. random variables distributed $N(\mu, \sigma^2)$ , with $\sigma^2 > 0$.

   (a) Shown that $\sqrt{n}(S - \sigma) \xrightarrow{D} N(0, \sigma^2/2)$.
   To do this first
   i) Find the expected value and covariance matrix of $\mathbf{Z}_i$ where $\mathbf{Z}_i' = (X_i, X_i^2)$.
   ii) State the multivariate central limit theorem giving the distribution of $\sqrt{n}(\bar{\mathbf{Z}} - \mu_Z)$, where $\mu_Z = E(Z_i)$, where $\bar{\mathbf{Z}}' = (\sum_i X_i/n, \sum_i X_i^2/n)$.
   Then
   iii) Show the desired result.

   (b) What is the limiting distribution of $\sqrt{n}(\bar{X} - \mu + S - \sigma)$? Justify your answer.

4. (10 PTS) Let $\Omega = \{1, 2, 3, 4\}$ and $A$ the $\sigma$-algebra of all subsets of $\Omega$. Let $B = \{\emptyset, \{1, 3\}, \{2, 4\}, \Omega\}$. What is the conditional expectation $Y = E[X|B]$ of the random variable $X(k) = k^2$?

5. (22 PTS) Let $X$ be a random variable with $E(X^2) < \infty$.

   (a) Show that $E(X) < \infty$.

   (b) State and prove Chebychev's inequality

   (c) Now state and prove the weak law of large numbers involving a sequence of iid random variables $X_1, X_2, \ldots$ with finite mean and variance.