Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. Passing standard: 70% with three problems essentially complete. Justify all your answers.

1. Let $M$ be a compact manifold and $X$ a vector field on $M$.
   
   (a) Prove that $X$ is complete.
   
   (b) Let $\{\varphi_t, t \in \mathbb{R}\}$ be the one-parameter group of diffeomorphisms associated to $X$ and let $f \in C^\infty(M)$. Prove that
   
   $$f \circ \varphi_1 - f \circ \varphi_0 = \int_0^1 \varphi_t^*(df)(X) \, dt.$$  

2. Let $\mathbb{P}^2$ denote two-dimensional, real projective space and set: $E = \mathbb{R}^3 \times \mathbb{P}^2 / \sim$, where $\sim$ is the equivalence relation
   
   $$(v, L) \sim (v', L') \quad \text{if and only if} \quad L = L' \text{ and } v' - v \in L.$$  

   Let $\pi: E \to \mathbb{P}^2$ be the restriction to $E$ of the second projection.
   
   (a) Prove that $\pi: E \to \mathbb{P}^2$ is a vector bundle.
   
   (b) How is $E$ related to the tautological bundle on $\mathbb{P}^2$?
   
   (c) Find a trivializing cover for $E$ and compute the transition matrices relative to that cover.

3. Let $p: S^n \to \mathbb{P}^n$ be the natural projection and $A: S^n \to S^n$ the antipodal map.
   
   (a) Prove that a $k$-form $\alpha$ on $S^n$ is the pullback $p^\ast \beta$ of a $k$-form $\beta$ on $\mathbb{P}^n$ if and only if $A^\ast \alpha = \alpha$.
   
   (b) Prove that $H^k_{dR}(\mathbb{P}^n) = 0$ for $0 < k < n$.
   
   (c) Prove that $H^n_{dR}(\mathbb{P}^n) = 0$ if $n$ is even and $H^n_{dR}(\mathbb{P}^n, \mathbb{R}) \cong \mathbb{R}$ if $n$ is odd.

4. Let $U \subset \mathbb{R}^3$ be open, and take a function $f \in C^\infty(U)$ without critical points. Consider the vector fields on $U$:

   $$X_1 = f_x \partial/\partial y - f_y \partial/\partial x,$$
   $$X_2 = f_y \partial/\partial z - f_z \partial/\partial y,$$
   $$X_3 = f_x \partial/\partial z - f_z \partial/\partial x.$$
Show that $X_1, X_2, X_3$ span a smooth distribution $D$ of rank two on $U$. Show that $D$ is integrable, and that $f$ is constant on any connected integral manifold of $D$.

5. Let $F: N \to M$ be a submersion of compact oriented manifolds.
   (a) Show how to give each fiber $F^{-1}(a)$ an orientation determined by the orientations of $N$ and $M$.
   (b) Suppose that $M$ is connected, and let $\alpha$ be a closed $(n-m)$-form, where $n = \dim N$, $m = \dim M$. Show that
   $$\int_{F^{-1}(a)} \alpha$$
   is independent of the point $a \in M$.

6. Let $\phi: M \to N$ be a smooth map between connected, oriented $n$-manifolds (without boundary). Prove that
   $$\left(\int_M \phi^* \alpha \right) \left(\int_N \beta \right) = \left(\int_N \alpha \right) \left(\int_M \phi^* \beta \right)$$
   for any $n$-forms $\alpha, \beta$ on $N$.

7. (1) Show that the real vector space $\mathbb{R}^3$ with the bilinear operation of the cross product forms a Lie algebra.
   (2) Give an example of a Lie group whose corresponding Lie algebra is isomorphic to the above one.

8. Two Riemannian metrics $g_1, g_2$ are called conformal equivalent if $g_2 = fg_1$ where $f$ is a smooth, positive function. Let $\phi: S^n \setminus \{(0,0, \cdots, 1)\} \to \mathbb{R}^n$ be the stereographic projection, and $h_0, g_0$ be the standard round metric on $S^n$ and flat metric on $\mathbb{R}^n$ respectively. Show that
   (i) $\phi$ is a diffeomorphism.
   (ii) $h_0$ and $\phi^*g_0$ are conformal equivalent.

9. Let $M = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, with the metric $g = y \, dx^2 + dy^2$, i.e.

   $$g(\partial/\partial x, \partial/\partial x) = y; \quad g(\partial/\partial y, \partial/\partial y) = 1; \quad g(\partial/\partial x, \partial/\partial y) = 0.$$

   (a) Compute the Gaussian curvature of $(M, g)$.
(b) Given that the Christoffel symbols of $g$ relative to the frame $\partial/\partial x, \partial/\partial y$ are given by:

$$
\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = \Gamma^2_{22} = 0 ; \quad \Gamma^2_{11} = -1/2 ; \quad \Gamma^1_{12} = 1/(2y),
$$

write the differential equations for a geodesic in $(M, g)$.

(c) Determine whether vertical or horizontal lines are geodesics and, if so, what is the appropriate parametrization.

10. Consider $\mathbb{R}^3$ with the product:

$$(x, y, z) \ast (x', y', z') := (x + x', e^x y' + y, e^x z' + z).$$

(a) Prove that $(\mathbb{R}^3, \ast)$ is a three-dimensional Lie group.

(b) Find a basis of left-invariant vector fields $X_1, X_2, X_3$ in $\mathbb{R}^3$ and compute the Lie brackets: $[X_i, X_j], 1 \leq i, j \leq 3$.

(c) Find a left-invariant Riemannian metric on $(\mathbb{R}^3, \ast)$. 