University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
For August 2012

Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

1. Let M be a compact manifold and X a vector field on M.

   (a) Prove that X is complete.

   (b) Let $\{\varphi_t, t \in \mathbb{R}\}$ be the one-parameter group of diffeomorphisms associated to X and let $f \in C^\infty(M)$. Prove that

   $$f \circ \varphi_1 - f \circ \varphi_0 \;=\; \int_0^1 \varphi_t^*(df)(X)\, dt\,.$$

2. Let $\mathbb{P}^2$ denote two-dimensional, real projective space and set: $E = \mathbb{R}^3 \times \mathbb{P}^2/\!\sim$, where $\sim$ is the equivalence relation

   $$(v, L) \sim (v', L') \quad \text{if and only if} \quad L = L' \text{ and } v' - v \in L.$$

   Let $\pi\colon E \to \mathbb{P}^2$ be the restriction to E of the second projection.

   (a) Prove that $\pi\colon E \to \mathbb{P}^2$ is a vector bundle.

   (b) How is E related to the tautological bundle on $\mathbb{P}^2$?

   (c) Find a trivializing cover for E and compute the transition matrices relative to that cover.

3. Let $p\colon S^n \to \mathbb{P}^n$ be the natural projection and $A\colon S^n \to S^n$ the antipodal map.

   (a) Prove that a k-form α on S^n is the pullback $p^*\beta$ of a k-form β on $\mathbb{P}^n$ if and only if $A^*\alpha = \alpha$.

   (b) Prove that $H^k_{dR}(\mathbb{P}^n) = 0$ for $0 < k < n$.

   (c) Prove that $H^n_{dR}(\mathbb{P}^n) = 0$ if n is even and $H^n_{DR}(\mathbb{P}^n, \mathbb{R}) \cong \mathbb{R}$ if n is odd.

4. Let $U \subset \mathbb{R}^3$ be open, and take a function $f \in C^\infty(U)$ without critical points. Consider the vector fields on U:

   $$\begin{aligned} X_1 &= f_x\partial/\partial y - f_y\partial/\partial x \\ X_2 &= f_y\partial/\partial z - f_z\partial/\partial y \\ X_3 &= f_x\partial/\partial z - f_z\partial/\partial x \end{aligned}$$

Show that X_1, X_2, X_3 span a smooth distribution D of rank two on U. Show that D is integrable, and that f is constant on any connected integral manifold of D.

5. Let $F\colon N \to M$ be a submersion of compact oriented manifolds.

   (a) Show how to give each fiber $F^{-1}(a)$ an orientation determined by the orientations of N and M.

   (b) Suppose that M is connected, and let α be a closed $(n-m)$-form, where $n = \dim N$, $m = \dim M$. Show that

   $$\int_{F^{-1}(a)} \alpha$$

   is independent of the point $a \in M$.

6. Let $\phi\colon M \to N$ be a smooth map between connected, oriented n-manifolds (without boundary). Prove that

   $$\left(\int_M \phi^*\alpha\right)\left(\int_N \beta\right) = \left(\int_N \alpha\right)\left(\int_M \phi^*\beta\right)$$

   for any n-forms α, β on N.

7. (1) Show that the real vector space $\mathbb{R}^3$ with the bilinear operation of the cross product forms a Lie algebra.

   (2) Give an example of a Lie group whose corresponding Lie algebra is isomorphic to the above one.

8. Two Riemannian metrics g_1, g_2 are called *conformal equivalent* if $g_2 = fg_1$ where f is a smooth, positive function. Let $\phi : S^n \setminus \{(0, 0, \cdots, 1)\} \to \mathbb{R}^n$ be the stereographic projection, and h_0, g_0 be the standard round metric on S^n and flat metric on $\mathbb{R}^n$ respectively. Show that

   (i) ϕ is a diffeomorphism.

   (ii) h_0 and $\phi^* g_0$ are conformal equivalent.

9. Let $M = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, with the metric $g = y\,dx^2 + dy^2$, i.e.

   $$g(\partial/\partial_x, \partial/\partial_x) = y\ ; \quad g(\partial/\partial_y, \partial/\partial_y) = 1\ ; \quad g(\partial/\partial_x, \partial/\partial_y) = 0.$$

   (a) Compute the Gaussian curvature of (M, g).

(b) Given that the Christoffel symbols of g relative to the frame ∂/∂_x, ∂/∂_y are given by:

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = \Gamma^2_{22} = 0 \; ; \quad \Gamma^2_{11} = -1/2 \; ; \quad \Gamma^1_{12} = 1/(2y)\,,$$

write the differential equations for a geodesic in (M, g).

(c) Determine whether vertical or horizontal lines are geodesics and, if so, what is the appropriate parametrization.

10. Consider $\mathbb{R}^3$ with the product:

$$(x, y, z) * (x', y', z') \;:=\; (x + x', e^x\, y' + y, e^x\, z' + z).$$

(a) Prove that $(\mathbb{R}^3, *)$ is a three-dimensional Lie group.

(b) Find a basis of left-invariant vector fields X_1, X_2, X_3 in $\mathbb{R}^3$ and compute the Lie brackets: $[X_i, X_j]$, $1 \leq i, j \leq 3$.

(c) Find a left-invariant Riemannian metric on $(\mathbb{R}^3, *)$.